ADJUNCTION AND INVERSION OF ADJUNCTION IN POSITIVE CHARACTERISTIC

by

Omprokash Das

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The dissertation of Omprokash Das
has been approved by the following supervisory committee members:

Christopher Hacon , Chair	03/02/2015 Date Approved
Aaron Bertram , Member	03/02/2015 Date Approved
Yuan-Pin Lee , Member	03/02/2015 Date Approved
Anurag Singh , Member	03/02/2015 Date Approved
Ilya Zharov , Member	03/02/2015 Date Approved

and by Peter Trapa , Chair of the Department of Mathematics

and by David B. Kieda, Dean of The Graduate School.

ABSTRACT

In this dissertation, we prove a characteristic $p > 0$ analogue of the log terminal inversion of adjunction and show the equality of the two technical terms F-Different and Different conjectured by Schwede. We also prove a special version of the (relative) Kawamata-Viehweg vanishing theorem for 3-folds, normality of minimal log canonical centers, Kodaira's Canonical Bundle formula for family of rational curves, and the Adjunction Formula on $\mathbb{Q}$-factorial 3-folds in characteristic $p > 5$.

To my father Late Umapati Das, mother Annapurna Das, and my wife Debosmita Gan Das

CONTENTS

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CHAPTER 1

INTRODUCTION

One of the main goals of algebraic geometry is to classify algebraic objects, namely algebraic varieties. In birational geometry, we want to classify algebraic varieties up to birational isomorphisms, i.e., we say that two algebraic varieties are "Birationally Equivalent" if their function fields are isomorphic. The Minimal Model Program (MMP) is in the heart of the birational classification of algebraic varieties. In higher dimension (dim ≥ 3), one needs to allow some singularities in order to run the MMP. Therefore, it is important to understand the singularities of the MMP. In the quest for understanding the MMP singularities, we find Adjunction and Inversion of Adjunction, two powerful technical tools in algebraic geometry which relate the singularities of a variety to the singularities of certain subvarieties. In this dissertation, we study these two technical tools in depth.

The Minimal Model Program in characteristic 0 depends heavily on the resolution of singularities and the Kawamata-Viehweg vanishing theorem. In characteristic $p > 0$, the first one is known only in dim ≤ 3 and the second one is known to fail (even in dimension 2). Recently in a series of papers by Hara, Hochster, Huneke, Mustaţá, Schwede, Smith, Tucker, and others, it has become clear that one can sometimes replace the vanishing theorems by use of test ideals, Frobenius maps, and the Serre vanishing theorem. The F-singularity techniques coming from the tight closure theory in commutative algebra have proved to be a powerful tool in studying birational geometry in characteristic $p > 0$. In this dissertation, we also study the adjunction and inversion of adjunction for F-singularities and their relations to the MMP singularities.

Chapter 1 contains mostly preliminary results. In this chapter, we define the F-singularities and the MMP singularities. We also state the known results on how they are related.

Chapter 2 is about Inversion of Adjunction. In this chapter, we state the known results on various statements of inversion of adjunction in characteristic 0 and p. We also prove one of our main theorems, namely the characteristic $p > 0$ analogue of Log Terminal inversion of adjunction in arbitrary dimension. More specifically, we prove the following theorem

Theorem A (Theorem 3.17, Corollary 6.5)**.** *Let $(X, S+B)$ be a pair where X is a normal variety, $S+B \geq 0$ is a $\mathbb{Q}$-divisor, K_X+S+B is $\mathbb{Q}$-Cartier and $S = \lfloor S+B \rfloor$ is reduced and irreducible. Let $\nu : S^n \to S$ be the normalization morphism, write $(K_X+S+B)|_{S^n} = K_{S^n}+B_{S^n}$. If (S^n, B_{S^n}) is strongly F-regular, then S is normal; furthermore, S is a unique center of sharp F-purity of $(X, S+B)$ in a neighborhood of S and $(X, S+B)$ is purely F-regular near S.*

Chapter 3 is about Vanishing theorems and Log Canonical centers. In this chapter, we prove some basic properties of LC centers for 3-folds in characteristic $p > 0$ which are already known in characteristic 0 and expected to hold in characteristic $p > 0$. Finally, we prove a special version of the Relative Kawamata-Viehweg vanishing theorem for $\mathbb{Q}$-factorial 3-folds in characteristic $p > 5$. Then using this theorem, we prove that the minimal LC centers of a 3-fold are normal in characteristic $p > 5$. More specifically, we prove the following theorems

Theorem B (Theorem 4.6)**.** *Let $(X, \Delta > 0)$ be a $\mathbb{Q}$-factorial 3-fold log canonical pair with isolated center W, $\operatorname{codim}_X W = 2$, and S a unique exceptional divisor dominating W with $a(S, X, \Delta) = -1$. Also assume that X has KLT singularities. Let $f : (Y, S+B) \to (X, W)$ be the corresponding divisorial extraction such that $K_Y+S+B = f^*(K_X+\Delta)$. Then $R^1 f_*\mathcal{O}_Y(-S) = 0$.*

Theorem C (Theorem 4.7)**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial 3-fold log canonical pair such that X has KLT singularities. If W is a minimal log canonical center of (X, Δ), then W is normal.*

Chapter 4 is about the adjunction formula on 3-folds in characteristic $p > 0$. In this chapter, we state the adjunction conjecture and various known cases of this conjecture. We prove a special version of Kodaira's Canonical Bundle formula for families of rational curves in characteristic $p > 0$. Finally, we prove the Adjunction Formula on $\mathbb{Q}$-factorial 3-folds in characteristic $p > 0$.

Theorem D (Theorem 5.13)**.** *Let $f : X \to Z$ be a proper surjective morphism, where X is a normal surface and Z is a smooth curve over an algebraically closed field k of char $(k) > 0$. Also assume that $Q = \sum_i Q_i$ is a divisor on Z such that f is smooth over $(Z - Supp(Q))$ with fibers isomorphic to $\mathbb{P}^1$. Let $D = \sum_j d_j P_j$ be a $\mathbb{Q}$-divisor on X, where $d_j = 0$ is allowed, which satisfies the following conditions:*

1. *$(X, D \geq 0)$ is KLT.*
2. *$D = D^h + D^v$, where $D^h = \sum_{f(D_j)=Z} d_j D_j$ and $D^v = \sum_{f(D_j) \neq Z} d_j D_j$. An irreducible component of D^h (resp. D^v) is called horizontal (resp. vertical) component.*
3. *$\mathrm{char}(k) = p > \frac{2}{\delta}$, where δ is the minimum non-zero coefficient of D^h.*
4. *$K_X + D \sim_{\mathbb{Q}} f^*(K_Z + M)$ for some $\mathbb{Q}$-Cartier divisor M on Z.*

Then there exists an effective $\mathbb{Q}$-divisor $D_{\mathrm{div}} \geq 0$ and a semi-ample $\mathbb{Q}$-divisor $D_{\mathrm{mod}} \geq 0$ on Z (as defined in 5.1.2) such that

$$K_X + D \sim_{\mathbb{Q}} f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

Theorem E (Theorem 5.14)**.** *Let $(X, D \geq 0)$ be a $\mathbb{Q}$-factorial 3-fold log canonical pair such that the coefficients of D are contained in a DCC set $I \subseteq [0, 1]$. Let W be a minimal log canonical center of (X, D), and codimension of W is 2. Also assume that X has KLT singularities and $\mathrm{char}(k) > \frac{2}{\delta}$, where δ is the non-zero minimum of the set $D(I)$ (defined in 5.1.1). Then the following hold:*

1. *W is normal.*
2. *There exists effective $\mathbb{Q}$-divisors D_W and M_W on W such that $(K_X + D)|_W \sim_{\mathbb{Q}} K_W + D_W + M_W$. Moreover, if $D = D' + D''$ with D' (resp. D'') the sum of all irreducible components which contain (resp. do not contain) W, then M_W is determined only by the pair (X, D').*

3. *There exists an effective $\mathbb{Q}$-divisor M'_W such that $M'_W \sim_{\mathbb{Q}} M_W$ and the pair $(W, D_W + M'_W)$ is KLT.*

Chapter 5 is about F-adjunction. In this chapter, we define the F-adjunction. The main theorem of this chapter is about the equality of the two technical terms F-Different, coming from F-adjunction, and the Different, coming from the usual adjunction. More specifically, we prove the following theorem

Theorem G (Theorem 6.4)**.** *Let $(X, S + \Delta \geq 0)$ be a pair, where X is a F-finite normal excellent scheme of pure dimension over a field k of characteristic $p > 0$ and $S + \Delta \geq 0$ is a $\mathbb{Q}$-divisor on X such that $(p^e - 1)(K_X + S + \Delta)$ is Cartier for some $e > 0$. Also assume that S is a reduced Weil divisor and $S \wedge \Delta = 0$. Then the F-Different, F-Diff$_{S^n}(\Delta)$ is equal to the Different, Diff$_{S^n}(\Delta)$, i.e., F-Diff$_{S^n}(\Delta)$ = Diff$_{S^n}(\Delta)$, where $S^n \to S$ is the normalization morphism.*

Notation and Conventions. We work over an algebraically closed field k of characteristic $p > 0$ throughout the whole dissertation, unless stated otherwise. We will use the standard notations from [Har77], [Laz04a], [Laz04b], and [KM98].

CHAPTER 2

F-SINGULARITIES AND MMP SINGULARITIES

In this chapter, we will define the F-singularities and the singularities of the Minimal Model Program (MMP). We will also compare the properties of these two types of singularities and state the know results about their relationship.

2.1 F-singularities

Definition 2.1. We say that a noetherian ring R of characteristic $p > 0$ is F-finite if F_*R is finitely generated as a R-module.

Definition 2.2. Let A be a normal domain with quotient field $K(A)$ and D, a $\mathbb{Q}$-Weil divisor on $X = \operatorname{Spec} A$. We define the A-module $A(D)$ as

$$A(D) = \{f \in K(A) : D + \operatorname{div}(f) \geq 0\} \cup \{0\}.$$

Definition 2.3. [HW02], [Har05], [HR76], [Tak04a], [Sch10]. Let A be a F-finite normal domain of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on $X = \operatorname{Spec} A$.

(1) We say that the pair (X, Δ) is strongly F-regular if for every non-zero $c \in A$, there exists $e > 0$ such that the composition

$$A \xrightarrow{F^e} F^e_* A \xrightarrow{F^e_*(c.\_)} F^e_* A \xhookrightarrow{\iota} F^e_* A(\lceil (p^e - 1)\Delta \rceil)$$

splits as a map of A-modules.

(2) (X, Δ) is purely F-regular if for every non-zero $c \in A$ which is not in any minimal prime ideal of $A(-\lfloor \Delta \rfloor) \subseteq A$, there exists $e > 0$ such that the composition

$$A \xrightarrow{F^e} F^e_* A \xrightarrow{F^e_*(c.\_)} F^e_* A \xhookrightarrow{\iota} F^e_* A(\lceil (p^e - 1)\Delta \rceil)$$

splits as a map of A-modules.

(3) (X, Δ) is sharply F-pure, if there exists an $e > 0$ such that the composition

$$A \xrightarrow{F^e} F^e_* A \xhookrightarrow{\iota} F^e_* A(\lceil (p^e - 1)\Delta \rceil)$$

splits as a map of A-modules.

Remark 2.4. Our definition of *purely F-regular* is the same as *divisorially F-regular* defined in [HW02].

Example 2.5.

1. It is obvious from the definition that 'Strongly F-regular $\Rightarrow$ Purely F-regular $\Rightarrow$ Sharply F-pure'.

2. Let $X = \operatorname{Spec} k[x]$ and $\Delta = 0$. Then $(X, 0)$ is strongly F-regular. Let $c = f(x) \in k[x] - \{0\}$. We have to show that there exists an $e > 0$ such that the map $\phi_e : k[x] \to F^e_* k[x]$ defined by $\phi_e(1) = f(x)$ as a $k[x]$-module splits. Let $\deg(f(x)) = n$. Now $F^e_* k[x]$ is a free $k[x]$ module with $\{1, x, x^2, \ldots, x^{p^e - 1}\}$ as a free basis over $k[x]$. Choose $e > n = \deg(f(x))$, then

$$B_e = \{1, x, x^2, \ldots, x^{n-1}, f(x), x^{n+1}, \ldots, x^{p^e-1}\}$$

is also a free basis of $F^e_* k[x]$ over $k[x]$. We define $\psi_e : F^e_* k[x] \to k[x]$ by $\psi_e(f(x)) = 1$ and $\psi_e(g(x)) = 0$ for all $g(x) \in B_e - \{f(x)\}$. Then ψ_e is the required splitting.

3. Let $X = \operatorname{Spec} k[x]$ and $\Delta = \operatorname{div}(x)$. Then (X, Δ) is purely F-regular but not strongly F-regular. First we will show that (X, Δ) is not strongly F-regular. Let $c = x \in k[x]$. We have to show that for any $e > 0$, the map $\phi_e : k[x] \to F^e_* k[x](\lceil (p^e - 1)\Delta \rceil) = F^e_* \left(k[x] \cdot \frac{1}{x^{p^e-1}} \right)$ defined by $\phi_e(1) = x$ does not split. On the contrary, assume that there exists an $e > 0$ such that ϕ_e splits, and $\psi_e : F^e_* \left(k[x] \cdot \frac{1}{x^{p^e-1}} \right) \to k[x]$ is the corresponding splitting. Then $1 = \psi_e(x) = \psi_e\left(\frac{x^{p^e}}{x^{p^e-1}} \right) = x \cdot \psi_e\left(\frac{1}{x^{p^e-1}} \right)$. This implies that x is a unit in $k[x]$, a contradiction. Now we will show that (X, Δ) is purely F-regular. We have $\lfloor \Delta \rfloor = \Delta = \operatorname{div}(x)$ and $k[x](-\lfloor \Delta \rfloor) = (x)$. Let $c = f(x) \in k[x]$ such that $f(0) \neq 0$. We have to show that there exists an $e > 0$ such that

the map $\phi_e : k[x] \to F^e_*\left(k[x]\cdot\frac{1}{x^{p^e-1}}\right)$ defined by $\phi_e(1) = f(x)$ splits. We see that $\left\{1, \frac{1}{x}, \ldots, \frac{1}{x^{p^e-1}}\right\}$ is a free basis of $F^e_*\left(k[x]\cdot\frac{1}{x^{p^e-1}}\right)$ over $k[x]$. Choose $e > \deg(f(x))$. Since $f(0) \neq 0$, $\left\{f(x), \frac{1}{x}, \frac{1}{x^2}, \ldots, \frac{1}{x^{p^e-1}}\right\}$ is also a free basis of $F^e_*\left(k[x]\cdot\frac{1}{x^{p^e-1}}\right)$ over $k[x]$. Thus we have a splitting ψ_e defined similarly as in Example (2) above.

4. Let $X = \operatorname{Spec} k[x,y]$ and $\Delta = \operatorname{div}(xy)$. Then (X,Δ) is sharply F-pure but neither strongly F-regular nor purely F-regular. First we will show that (X,Δ) is not purely F-regular. Let $c = x+y \in k[x,y]$ and consider the map $\phi_e : k[x,y] \to F^e_*\left(k[x,y]\cdot\frac{1}{x^{p^e-1}y^{p^e-1}}\right)$ defined by $\phi_e(1) = x+y$. On the contrary, assume that $\psi_e : F^e_*\left(k[x,y]\cdot\frac{1}{x^{p^e-1}y^{p^e-1}}\right) \to k[x,y]$ is a splitting of ϕ_e. Then $1 = \psi_e(x+y) = x\cdot\psi_e\left(\frac{y^{p^e-1}}{x^{p^e-1}y^{p^e-1}}\right) + y\cdot\psi_e\left(\frac{x^{p^e-1}}{x^{p^e-1}y^{p^e-1}}\right)$, which is a contradiction to the fact that $(x,y) \neq k[x,y]$. Now we will show that (X,Δ) is sharply F-pure. We have to show that the map $\phi_e : k[x,y] \to F^e_*\left(k[x,y]\cdot\frac{1}{x^{p^e-1}y^{p^e-1}}\right)$ defined by $\phi_e(1) = 1$ splits. Obviously $B_e = \left\{\frac{x^{\lambda_1}y^{\lambda_2}}{x^{p^e-1}y^{p^e-1}} : 0 \leq \lambda_1, \lambda_2 \leq p^e-1\right\}$ is a free basis of F^e_*R, where $R = \left(k[x,y]\cdot\frac{1}{x^{p^e-1}y^{p^e-1}}\right)$ over $k[x,y]$. Since $1 \in B_e$, we can define $\psi_e : F^e_*\left(k[x,y]\cdot\frac{1}{x^{p^e-1}y^{p^e-1}}\right) \to k[x,y]$ by $\psi_e(1) = 1$ and $\psi_e(g(x,y)) = 0$ for all $g(x,y) \in B_e - \{1\}$. Clearly ψ_e is a splitting of ϕ_e.

5. Let $X = \operatorname{Spec} k[x]$ and $\Delta = \operatorname{div}(x^2)$. Then (X,Δ) is not sharply F-pure. We will show that there is no $e > 0$ such that the map $\phi_e : k[x] \to F^e_*R$, where $R = \left(k[x]\cdot\frac{1}{x^{2p^e-2}}\right)$, defined by $\phi_e(1) = 1$ splits. On the contrary, assume that there exists an $e > 0$ such that $\psi_e : F^e_*\left(k[x]\cdot\frac{1}{x^{2p^e-2}}\right) \to k[x]$ gives a splitting of ϕ_e. Then $1 = \psi_e(1) = x\cdot\phi_e\left(\frac{x^{p^e-2}}{x^{2p^e-2}}\right)$. This implies that x is a unit in $k[x]$, a contradiction.

Definition 2.6. Let $(X, \Delta \geq 0)$ be a pair where X is a normal variety and $L_{g,\Delta} = (1-p^g)(K_X+\Delta)$ an integral Cartier divisor for some $g > 0$. Then by the *Grothendieck Trace map*, we get a morphism

$$\phi^g : F^g_*\mathcal{O}_X(L_{g,\Delta}) \to \mathcal{O}_X.$$

Following [Pat12], we define the *non-F-pure ideal* $\sigma(X, \Delta)$ of (X, Δ) to be:

$$\sigma(X, \Delta) = \bigcap_{e \geq 0} \phi^{eg} \left(F_*^{eg} \mathcal{O}_X(L_{eg,\Delta}) \right).$$

Remark 2.7. The above intersection is a descending intersection. By [Sch14, Remark 2.9], this intersection stabilizes, i.e.,

$$\phi^{eg} \left(F_*^{eg} \mathcal{O}_X(L_{eg,\Delta}) \right) = \sigma(X, \Delta) \text{ for all } e \gg 0.$$

Remark 2.8. If $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p, then $(X, \Delta \geq 0)$ is *sharply F-pure* if and only if $\sigma(X, \Delta) = \mathcal{O}_X$.

2.2 Singularities of the Minimal Model Program (MMP)

Let X be a normal variety and $f : Y \to X$ be birational morphism. Any prime Weil divisor $E \subseteq Y$ is called a *divisor over* X. The closure of its image $\overline{f(E)}$ is called the *center* of E in X.

Let (X, Δ) be a pair, where X is a normal variety and Δ a $\mathbb{Q}$-divisor (not necessarily effective) such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a proper birational morphism. Then we write

$$K_Y = f^*(K_X + \Delta) + \sum a(E_i, X, \Delta) E_i,$$

where $f_*(K_Y) = K_X$ and the sum runs over all prime Weil divisors in Y.
The rational numbers $a(E_i, X, \Delta)$ are called the *discrepancy* of the divisor $E_i \subseteq Y$. If the center of E_i is a component $\Delta = \sum d_i D_i$, say D_i, then $a(E_i, X, \Delta) := -d_i$. If E_i is not an exceptional divisor of f and its center is not a component of Δ, then we define $a(E_i, X, \Delta) := 0$. Thus the above sum is a finite sum.

$\Delta = \sum d_i D_i$ is called a *boundary* divisor if $0 \leq d_i \leq 1$ for all i, and a *subboundary* if $d_i \leq 1$ for all i.

Definition 2.9. The *discrepancy* of (X, Δ) is defined by

$$\text{discrep}(X, \Delta) := \inf \left\{ a(E, X, \Delta) : E \text{ is an exceptional divisor over } X \right\}.$$

Definition 2.10. The *total discrepancy* of (X, Δ) is defined by

$$\text{totaldiscrep}(X, \Delta) := \inf \{ a(E, X, \Delta) : E \text{ is a divisor over } X \}.$$

Definition 2.11. Let (X, Δ) be a pair, where X is a normal variety, $\Delta = \sum d_i D_i$ is a $\mathbb{Q}$-divisor such that $d_i \leq 1$ for all i, and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that (X, Δ) is

Terminal	if $\mathrm{discrep}(X, \Delta) > 0$.
Canonical	if $\mathrm{discrep}(X, \Delta) \geq 0$.
Kawamata Log Terminal (KLT)	if $\mathrm{totaldiscrep}(X, \Delta) > -1$.
Purely Log Terminal (PLT)	if $\mathrm{discrep}(X, \Delta) > -1$.
Log Canonical (LC)	if $\mathrm{totaldiscrep}(X, \Delta) \geq -1$.

Definition 2.12. Let (X, Δ) be a pair, where X is a normal variety, $\Delta = \sum d_i D_i$ is a $\mathbb{Q}$-divisor such that $d_i \leq 1$ for all i, and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say (X, Δ) is Divisorially Log Terminal (DLT) if there exists a closed subset $Z \subseteq X$ such that

1. $X \backslash Z$ is smooth and $\Delta|_{X \backslash Z}$ is a SNC divisor.
2. If $f : Y \to X$ is a proper birational morphism and E is a prime divisor on Y such that $\mathrm{center}_X E \subseteq Z$, then $a(E, X, \Delta) > -1$.

Definition 2.13. Let (X, Δ) be a pair, where X is a normal variety and Δ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a proper birational morphism and $E \subseteq Y$ a prime divisor of Y. If the discrepancy $a(E, X, \Delta) = -1$ and $f(E) = W$, then W is called a Non-Kawamata Log Terminal Center (NKLT Center) or a Log Canonical Center (LC Center) of (X, Δ), and E is a Log Canonical Place of W. If $a(E, X, \Delta) < -1$, then W is called a Non-Log Canonical Center (NLC Center) of (X, Δ).

Example 2.14.

1. If $\dim X \geq 2$, then each of the singularities in the list of Definition 2.11 implies the next one, except canonical does not imply KLT when Δ has a component with coefficient 1, e.g., $(\mathbb{A}^2, \mathrm{div}(x))$ is canonical but not KLT. Also DLT implies LC and is implied by PLT.
2. $(\mathbb{A}^2, 0)$ is terminal.
3. $(\mathbb{A}^2, \mathrm{div}(x))$ is canonical but not terminal.

4. $(\mathbb{A}^2, \frac{1}{2}\operatorname{div}(x) + \frac{3}{4}\operatorname{div}(y))$ is KLT but not canonical.

5. $(\mathbb{A}^2, \operatorname{div}(x) + \operatorname{div}(y))$ is DLT but not PLT.

6. $(\mathbb{A}^2, \frac{5}{6}\operatorname{div}(y^2 - x^3))$ is LC but not DLT.

For other properties of these singularities, see [Kol96], [KM98] and [Kol13].

2.3 F-singularities vs. MMP singularities

The following facts suggest that the strongly F-regular, purely F-regular, and sharply F-pure are the characteristic $p > 0$ analogue of KLT, PLT, and LC singularities, respectively.

If X is a smooth curve, then

1. (X, Δ) is strongly F-regular if and only if (X, Δ) is KLT.

2. (X, Δ) is purely F-regular if and only if (X, Δ) is PLT.

3. (X, Δ) is sharply F-pure if and only if (X, Δ) is LC.

In higher dimension, the following implications hold.

Theorem 2.15. *[HW02, Theorem 3.3]*

1. *If (X, Δ) is strongly F-regular, then (X, Δ) is KLT.*

2. *If (X, Δ) is purely F-regular, then (X, Δ) is PLT.*

3. *If (X, Δ) is sharply F-pure, then (X, Δ) is LC.*

The converse results fail in each case in higher dimensions. For example, consider $(\mathbb{A}^2, (\frac{5}{6} - \epsilon)C)$, where C is the cusp $y^2 - x^3 = 0$ and $0 < \epsilon < \frac{5}{6}$. Then $(\mathbb{A}^2, (\frac{5}{6} - \epsilon)C)$ is KLT but not sharply F-pure in characteristic p, for $p \equiv 5 \pmod 6$ and $0 < \epsilon < \frac{1}{6p}$.

In [Har98], Hara showed that a partial converse of the Part (1) of Theorem 2.15 holds in dimension 2. More precisely, he proved that if $\Delta = 0$, $\dim X = 2$, and $\operatorname{char}(k) > 5$, then $(X, 0)$ is KLT if and only if $(X, 0)$ is strongly F-regular (see [Har98, Corollary 4.9]). In a recent article by Hacon and Xu [HX13], they extended Hara's result to the case when the coefficients of Δ belong to the set $\{1 - \frac{1}{n} : n \geq 1\} \cup \{1\}$ (see [HX13, Theorem 3.1]). In another recent article by Cascini, Gongyo, and Schwede [CGS14], they extended Hara's result to the case when the coefficients of Δ are in a fixed DCC set and $\mathrm{char}(k) > I_0$, where I_0 is completely determined by the coefficients of Δ (see [CGS14, Theorem 1.1]).

For the following two theorems we will assume that (X, Δ) is a pair in characteristic 0 and (X_p, Δ_p) is the reduction of (X, Δ) modulo the prime p.

Theorem 2.16. *[HW02, Theorem 3.7]*

1. *If (X_p, Δ_p) is strongly F-regular for infinitely many p, then (X, Δ) is KLT.*
2. *If (X_p, Δ_p) is purely F-regular for infinitely many p, then (X, Δ) is PLT.*
3. *If (X_p, Δ_p) is sharply F-pure for infinitely many p, then (X, Δ) is LC.*

Theorem 2.17. *[Tak04b, Corollary 3.4][Tak08, Corollary 5.4]*

1. *If (X, Δ) is KLT, then (X_p, Δ_p) is strongly F-regular for all $p \gg 0$.*
2. *If (X, Δ) is PLT, then (X_p, Δ_p) is purely F-regular for all $p \gg 0$.*

2.4 What Is Adjunction and Inversion of Adjunction?

Let $(X, \Delta \geq 0)$ be a pair, where X is a normal variety and Δ is an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Adjunction in general means that if we assume something about (X, Δ), then we can conclude something about a LC center of (X, Δ). Inversion of Adjunction on the other hand means that if we assume something about a LC center of (X, Δ), then we can conclude something about the pair (X, Δ). Inversion of adjunction is in general harder to prove than adjunction.

If X is a smooth variety and S is smooth prime divisor of X, then we know that $(K_X + S)|_S \sim K_S$. This formula is known as the *adjunction formula*. If X is singular, then the above formula needs a correction term called the 'Different'.

The adjunction formula for higher codimensional subvarieties is mostly conjectural. However, many partial results are known in characteristic 0. In this dissertation, we prove a special version of this formula for 3-folds in characteristic $p > 5$.

2.4.1 Example

Let X be a cone over a rational curve C of degree n. Let $f : Y \to X$ be the blowup of X at the vertex. If E is the exceptional divisor of f, then $E \cong C$ and $E^2 = -n$ (see [Har77, Chapter V, Example 2.11.4]). Let S be a ruling of X and S', the strict transform of S. Then we have

$$f^*S = \frac{1}{n}E + S' \text{ and } K_Y = f^*K_X + \left(\frac{2}{n} - 1\right)E.$$

Thus

$$K_Y + S' + \left(1 - \frac{1}{n}\right)E = f^*(K_X + S).$$

Since $f|_{S'} : S' \to S$ is an isomorphism, we get

$$(K_X + S)|_S = f^*(K_X + S)|_{S'} = \left(K_Y + S' + \left(1 - \frac{1}{n}\right)E\right)\Big|_{S'} \sim K_{S'} + \left(1 - \frac{1}{n}\right)P,$$

where $P = S' \cap E$.

If $f(P) = Q$, then Q is the vertex of the cone X and the divisor $\left(1 - \frac{1}{n}\right)Q$ is the 'Different' of the adjunction formula for the pair (X, S). Observe that (X, S) has PLT singularities for $n > 1$, thus by adjunction, we can conclude that $\left(S, \left(1 - \frac{1}{n}\right)Q\right)$ has KLT singularities (see Theorem 3.1).

To use inversion of adjunction, first we apply the adjunction formula on a LC center, say W, of (X, Δ) to get a pair (W, Δ_W). Now if we know some information about the singularities of the pair (W, Δ_W), then by inversion of adjunction we can make some conclusion about the singularities of the pair (X, Δ) in a neighborhood of W. This allows us to extract information from lower dimension to higher dimension.

2.4.2 Resolution of Singularities

After [Abh65] and [Hir84], we know that the resolution of singularities exists for *excellent surfaces* in characteristic $p > 0$; see also [Lip78]. We will also use the existence of *minimal resolution*.

Theorem 2.18 (Existence of minimal resolution)**.** *Let X be an excellent surface. Then there exists a unique resolution $f : Y \to X$, i.e., f is a proper birational morphism and Y is non-singular, such that any other resolution $g : Z \to X$ of X factors through f.*

Proof. For a proof, see [Lip69, 27.3]. Also consult [Lip78], [Kol13, 2.25], and [Kol07, 2.16]. □

Remark 2.19. The regular surface Y in the theorem above is an excellent surface and not necessarily a variety. Also Y does not contain any (-1)-curves over X and K_Y is *nef* relative to X.

We will use the following properties of Weil divisors and reflexive sheaves throughout this article. For the convenience of the reader, we record some useful properties of reflexive sheaves that we will use without comment.

Proposition 2.20. *[Har77] and [Har94, Proposition 1.11, Theorem 1.12] Let $X = \operatorname{Spec} R$ be a normal affine variety and M and N finitely generated R-modules. Then*
(1) *M is reflexive if and only if M is S_2.* (2) *$Hom_R(M, R) = M^\vee$ is reflexive.* (3) *If R is of characteristic $p > 0$ and F-finite (see Definition* (2.1)*), then M is reflexive if and only if F^e_*M is reflexive, where $F^e : X \to X$ be the e-iterated Frobenius morphism.* (4) *If N is reflexive, then $Hom_R(M, N)$ is also reflexive.* (5) *Suppose M is reflexive and $Z \subseteq X$ be a closed subset of codimension* 2*. Set $U = X - Z$ and let $i : U \to X$ be the inclusion. Then $i_*(M|_U) \cong M^{\vee\vee} \cong M$.* (6) *With the notations as in* (5)*, the restriction map to U induces an equivalence of categories from reflexive coherent sheaves on X to the reflexive coherent sheaves on U.* (7) *If $f : \mathscr{F} \to \mathscr{G}$ is a morphism between coherent sheaves on X, then there exists a natural morphism $f' : \mathscr{F}^{\vee\vee} \to \mathscr{G}^{\vee\vee}$ such that $f'|_U = f|_U$ for some open set $U \subseteq X$. In particular, if $\mathscr{G}$ is reflexive, i.e., $\mathscr{G} = \mathscr{G}^{\vee\vee}$, then $f : \mathscr{F} \to \mathscr{G}$ factors through $f' : \mathscr{F}^{\vee\vee} \to \mathscr{G}$.*

Proposition 2.21. *[Har94, Proposition 2.9] and [Har07, Remark 2.9] Let X be a normal variety and D be a Weil divisor on X. Then there is a one-to-one correspondence between the effective divisors linearly equivalent to D and the non-zero sections $s \in \Gamma(X, \mathcal{O}_X(D))$ modulo multiplication by units in $H^0(X, \mathcal{O}_X)$.*

CHAPTER 3

INVERSION OF ADJUNCTION

Inversion of adjunction is a powerful tool in studying the birational geometry of algebraic varieties. It has been studied heavily in characteristic 0. It is well known that if $(X, S+B)$ is a pair where $\lfloor S+B \rfloor = S$ is irreducible and reduced, then $(X, S+B)$ is PLT near S if and only if (S^n, B_{S^n}) is KLT, where $S^n \to S$ is the normalization of S and $K_{S^n} + B_{S^n} = (K_X + S + B)|_{S^n}$ is defined by adjunction. The proof follows from the resolution of singularities and the relative Kawamata-Viehweg vanishing theorem. In characteristic $p > 0$ and in higher dimensions (dim > 3), existence of the resolution of singularities is not known and the vanishing theorem (Kawamata–Viehweg) is known to fail, so we cannot expect a similar proof here. In this chapter, we prove a characteristic $p > 0$ analog of the *'Log terminal inversion of adjunction'* mentioned above.

The following results on inversion of adjunction are known in characteristic 0 in arbitrary dimension.

Theorem 3.1. *[KM98, Theorem 5.50, Theorem 5.51][Kaw97] Let $(X, S+B \geq 0)$ be a pair in characteristic 0 where X is a normal variety, $S+B \geq 0$ is a $\mathbb{Q}$-divisor such that $S \wedge B = \emptyset$ and $\lfloor S+B \rfloor = S$ is a reduced Weil divisor, and $K_X + S + B$ is $\mathbb{Q}$-Cartier. Let $B_{S^n} \geq 0$ be a $\mathbb{Q}$-divisor on the normalization S^n of S defined by the adjunction formula $(K_X + S + B)|_{S^n} \sim_{\mathbb{Q}} K_{S^n} + B_{S^n}$. Then*

1. *$(X, S+B)$ is PLT on a neighborhood of S if and only if (S^n, B_{S^n}) is KLT. Moreover if $(X, S+B)$ is PLT, then S is normal.*

2. *$(X, S+B)$ is LC on a neighborhood of S if and only if (S^n, B_{S^n}) is LC.*

The following versions of inversion of adjunction in characteristic 0 for LC centers of arbitrary codimensions are known due to [Tak04a], [Eis11], [Hac14].

Theorem 3.2. *[Tak04a, Theorem 4.1, 4.2] Let X be a non-singular variety over a field of characteristic 0 and $Y = \sum_{i=1}^{k} t_i Y_i$ a formal combination where $t_i > 0$ are real numbers and $Y_i \subsetneq X$ are closed subschemes. Let $Z \subsetneq X$ be a normal $\mathbb{Q}$-Gorenstein closed subvariety such that $Z \not\subseteq \cup_{i=1}^{k} Y_i$. Then*

1. *If $(Z, Y|_Z)$ is LC, then $(X, Y + Z)$ is LC on a neighborhood of Z.*

2. *If $(Z, Y_| Z)$ is KLT, then $(X, Y + Z)$ is PLT near Z.*

Theorem 3.3. *[Eis11, Corollary 4.1] Let $(X, \Delta \geq 0)$ be a pair, where X is a smooth complex projective variety and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor. Let Z be an exceptional LC center of (X, Δ) in a neighborhood of the generic point of Z. Then any generic Kawamata-Different is KLT on the normalization Z^n if and only if (X, Δ) is LC and Z is a minimal LC center of (X, Δ).*

Theorem 3.4. *[Hac14] Let V be a LC center of a pair $(X, \Delta = \sum \delta_i \Delta_i)$, where X is a normal variety in characteristic 0, Δ is a $\mathbb{Q}$-divisor, $0 \leq \delta_i \leq 1$, and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then (X, Δ) is log canonical on a neighborhood of V if and only if $(V^n, \mathbf{B}(V; X, \Delta))$ is log canonical.*

The following results on inversion of adjunction are known in characteristic $p > 0$.

Theorem 3.5. *[HW02, Theorem 4.9] Let X be a $\mathbb{Q}$-Gorenstein normal variety with $\operatorname{char}(k) \nmid$ index(K_X) and S a Cartier divisor. If $(S, 0)$ is strongly F-regular, then (X, S) is purely F-regular near S.*

Theorem 3.6. *[HX13, Theorem 6.2][Bir13] Let $(X, S + B \geq 0)$ be pair where X is a 3-fold normal variety in characteristic $p > 5$, $S + B \geq 0$ is a $\mathbb{Q}$-divisor such that $\lfloor S + B \rfloor = S$ is a prime Weil divisor, and $K_X + S + B$ is $\mathbb{Q}$-Cartier. Then*

1. *$(X, S + B)$ is LC on a neighborhood of S if and only if (S^n, B_{S^n}) is LC, where $S^n \to S$ is the normalization and $K_{S^n} + B_{S^n} = (K_X + S + B)|_{S^n}$.*

2. *If X is a $\mathbb{Q}$-factorial, then $(X, S + B)$ is PLT on a neighborhood of S if and only if (S^n, B_{S^n}) is KLT. Moreover if $(X, S + B)$ is PLT, then S is normal.*

In characteristic $p > 0$, analogous results for F-singularities are known in arbitrary dimension due to Schwede [Sch09].

Theorem 3.7. *[Sch09, Theorem 5.2] Let $(X, \Delta \geq 0)$ be a pair, where X is a normal variety in characteristic $p > 0$, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor, and $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p. Let $W \subseteq X$ be a normal subvariety of X which is a sharp F-pure center of (X, Δ). Also assume that $\Delta_W \geq 0$ is a $\mathbb{Q}$-divisor on W defined by F-adjunction. Then*

1. *(X, Δ) is sharply F-pure on a neighborhood of W if and only (W, Δ_W) is sharply F-pure.*
2. *W is a minimal F-pure center of (X, Δ) if and only if (W, Δ_W) is strongly F-regular.*

The log terminal inversion of adjunction for surfaces was known for a long time in characteristic $p > 0$, it follows from the exact same proof of the characteristic 0 case, since the resolution of singularities exists for surfaces in characteristic $p > 0$ and also the relative Kawamata-Viehweg vanishing theorem holds. In [HX13, 4.1], Hacon and Xu proved the Theorem 3.17 using the resolution of singularities, so in particular, their proof establishes the result for $\dim X \leq 3$. Our first proof of the Theorem 3.17 closely follows the techniques used in [HX13].

3.1 Some Lemmas and Propositions

We will need the following lemmas and propositions to prove the main theorem of this chapter (Theorem 3.17).

Lemma 3.8. *Let $(X, \Delta \geq 0)$ be a pair, where X is a normal excellent surface and $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Let $f : (Y, D) \to (X, \Delta)$ be a log resolution where $K_Y + D = f^*(K_X + \Delta)$. Write $D = \sum d_i D_i$, $A = \sum_{i:d_i<1} d_i D_i$ and $F = \sum_{i:d_i \geq 1} d_i D_i$. Then Supp $F =$ Supp $\lfloor F \rfloor$ is connected in a neighborhood of any fiber of f.*

Proof. By definition

$$\begin{aligned}\lceil -A \rceil - \lfloor F \rfloor = K_Y - (K_Y + D) + \{A\} + \{F\} = K_Y + (-(K_Y + D) + f_*^{-1}(\{\Delta\}))\\ + (\{A\} - f_*^{-1}(\{\Delta\})) + \{F\}.\end{aligned}$$

Now $\lceil -A \rceil - \lfloor F \rfloor$ is an integral Cartier divisor and $-(K_Y + D) + f_*^{-1}(\{\Delta\}) \equiv_f f_*^{-1}(\{\Delta\})$ is f-nef; therefore, by [KK, 2.2.5] (also see [Kol13, 10.4]), we have

$$R^1 f_* \mathcal{O}_Y(\lceil -A \rceil - \lfloor F \rfloor) = 0.$$

Applying f_* to the exact sequence

$$0 \to \mathcal{O}_Y(\lceil -A \rceil - \lfloor F \rfloor) \to \mathcal{O}_Y(\lceil -A \rceil) \to \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil) \to 0$$

we obtain that

$$f_* \mathcal{O}_Y(\lceil -A \rceil) \to f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil) \text{ is surjective.} \tag{3.1}$$

Since $\lceil -A \rceil$ is f-exceptional and effective, $f_* \mathcal{O}_Y(\lceil -A \rceil) = \mathcal{O}_X$. Suppose by contradiction that $\lfloor F \rfloor$ has at least two connected components $\lfloor F \rfloor = F_1 \cup F_2$ in a neighborhood of $g^{-1}(x)$ for some $x \in X$. Then

$$f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil)_{(x)} \cong f_* \mathcal{O}_{F_1}(\lceil -A \rceil)_{(x)} \oplus f_* \mathcal{O}_{F_2}(\lceil -A \rceil)_{(x)},$$

and neither of these summands is zero. Thus $f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil)_{(x)}$ cannot be the quotient of $\mathcal{O}_{x,X} \cong f_* \mathcal{O}_Y(\lceil -A \rceil)_{(x)}$, a contradiction. $\square$

Corollary 3.9. *Let $(X, S + B \geq 0)$ be a pair such that X is a normal excellent scheme of dimension n, $K_X + S + B$ is $\mathbb{Q}$-Cartier, and $\lfloor S + B \rfloor = S$ is reduced and irreducible. Further assume that $\nu : S^n \to S$ is the normalization of S and (S^n, B_{S^n}) is KLT, where $K_{S^n} + B_{S^n} = (K_X + S + B)|_{S^n}$. Then S is normal in codimension* 1.

Proof. Let $\mathfrak{p} \in X$ be a codimension 2 point of X contained in S, $X_\mathfrak{p} = \operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}}$ and $D_\mathfrak{p} = S_\mathfrak{p} + B_\mathfrak{p}$ the restriction of $S + B$ to $X_\mathfrak{p}$. Further assume that $g : (X', D') \to (X_\mathfrak{p}, D_\mathfrak{p})$ is a log resolution and let

$$K_{X'} + D' = g^*(K_{X_\mathfrak{p}} + D_\mathfrak{p}). \tag{3.2}$$

Let T be the strict transform of $S_\mathfrak{p}$, then restricting both sides of the above equation to T, we get

$$K_T + (D' - T)|_T = u^*(K_{S^n_\mathfrak{p}} + B_{S^n_\mathfrak{p}}) \tag{3.3}$$

where $u : T \to S^n_\mathfrak{p}$ is the induced morphism.

Let $A = \sum_{i:d_i<1} d_i D'_i$ and $F = \sum_{i:d_i\geq 1} d_i D'_i$ be as in the lemma above where $A + F = D'$.

Since (S^n, B_{S^n}) is KLT, from (3.3) we get $\lfloor (D' - T)|_T \rfloor \leq 0$. Thus if $\lfloor F \rfloor$ has another component say T_1, then $T \cap T_1 = \emptyset$, but $g(T) \cap g(T_1) \neq \emptyset$, which is a contradiction by Lemma 3.8. Hence $\lfloor F \rfloor = T$.
Now from (3.1) we get that

$$\mathcal{O}_{X_\mathfrak{p}} \to g_*\mathcal{O}_T(\lceil -A \rceil) \text{ is surjective.}$$

But this map factors through $\mathcal{O}_{S_\mathfrak{p}}$ and $g_*\mathcal{O}_T(\lceil -A \rceil)$ contains $\nu_*\mathcal{O}_{S^n_\mathfrak{p}}$, where $\nu : S^n \to S$ is the normalization morphism, hence $\mathcal{O}_{S_\mathfrak{p}} \to \nu_*\mathcal{O}_{S^n_\mathfrak{p}}$ is surjective and so $S_\mathfrak{p} = S^n_\mathfrak{p}$. □

Lemma 3.10. *With notations as in the proof of Corollary 3.9 above, assume that $(S^n_\mathfrak{p}, B_{S^n_\mathfrak{p}})$ is KLT, then $(X_\mathfrak{p}, S_\mathfrak{p} + B_\mathfrak{p})$ is PLT.*

Proof. Rewriting (3.3) as below

$$K_T = u^*(K_{S^n_\mathfrak{p}} + B_{S^n_\mathfrak{p}}) - (A + F')|_T$$

where $F' = F - T$, we see that $(X_\mathfrak{p}, S_\mathfrak{p} + B_\mathfrak{p})$ is PLT if and only if $F' \cap f^{-1}(S_\mathfrak{p}) = \emptyset$ or equivalently $F' \cap f^{-1}(\mathfrak{p}) = \emptyset$. Now $(S^n_\mathfrak{p}, B_{S^n_\mathfrak{p}})$ is KLT, so $F' \cap T = \emptyset$, therefore, by Lemma 3.8, it follows that $F' \cap f^{-1}(\mathfrak{p}) = \emptyset$, this completes the proof. □

Proposition 3.11. *With the same notations as in Lemma 3.10, if $(X_\mathfrak{p}, S_\mathfrak{p} + B_\mathfrak{p})$ is PLT, then $X_\mathfrak{p}$ is $\mathbb{Q}$-factorial. In particular, for each Weil divisor D on X, there is an open set $U \subseteq X$ (depending on D) containing all codimension 1 points of S, i.e., $\mathrm{codim}_S(S - U) \geq 2$ such that $D|_U$ is $\mathbb{Q}$-Cartier.*

Proof. Since $(X_\mathfrak{p}, S_\mathfrak{p} + B_\mathfrak{p})$ is PLT, $(X_\mathfrak{p}, 0)$ is numerically KLT, by [KM98, Corollary 4.2]. Let $f : Y \to X_\mathfrak{p}$ be the minimal resolution of $X_\mathfrak{p}$ and Δ_Y, the f-exceptional $\mathbb{Q}$-divisor satisfying the following relation as in [KM98, 4.1]:

$$K_Y + \Delta_Y \equiv_f 0.$$

Since K_Y is nef, Δ_Y is effective by the Negativity lemma. Also, the coefficients of Δ_Y are strictly less then 1, since $(X_\mathfrak{p}, 0)$ is numerically KLT. Therefore, $\lfloor \Delta_Y \rfloor = 0$. Then by [FT12, Theorem 6.2 (2)], $R^1 f_* \mathcal{O}_X = 0$. Hence $X_\mathfrak{p}$ is a rational surface. Then by [Lip69, 17.1], the Weil divisor class group $\mathrm{WDiv}(X_\mathfrak{p})$ of $X_\mathfrak{p}$ is finite. In particular, $X_\mathfrak{p}$ is $\mathbb{Q}$-factorial. $\square$

Proposition 3.12. *Let $X = \operatorname{Spec} A$ be an algebraic variety and $S = \operatorname{Spec} A/\mathfrak{p}$ be a prime Weil divisor on X. Then there exists a normal variety Y and a projective birational morphism $f : Y \to X$ such that the strict transform S' of S is the normalization of S.*

Proof. Let $\nu : S^n \to S$ be the normalization of S. Since ν is proper and birational and S is quasi-projective, it is given by a blow up of an ideal of A containing $\mathfrak{p}$. Let I be the corresponding ideal in A. Blowing up X along the ideal I, we get the following commutative diagram:

$$\begin{tikzcd} S^n \arrow[r, hook] \arrow[d, "\nu"'] & Y_1 \arrow[d, "f_1"] \\ S \arrow[r, hook] & X \end{tikzcd}$$

where $Y_1 = \operatorname{Proj} \oplus_{d \geq 0} I^d$ and $f_1 : Y_1 \to X$ is the blow up morphism.

Observe that there are open affine sets $X^\circ \subseteq X_{\text{smooth}}$ and $S^\circ \subseteq S_{\text{smooth}}$ such that $S^\circ = X^\circ \cap S$. Let $\pi : Y \to Y_1$ be the normalization morphism of Y_1, and S', the strict transform of S^n under π. Then we have the following commutative diagram:

$$\begin{tikzcd} S' \arrow[r, hook] \arrow[d, "\pi|_{S'}"'] & Y \arrow[d, "\pi"] \\ S^n \arrow[r, hook] & Y_1 \end{tikzcd}$$

Now $\pi|_{S'} : S' \to S^n$ is a finite birational morphism between two varieties with S^n normal, hence it is an isomorphism; in particular, S' is normal. Set $f = f_1 \circ \pi$, then $f : Y \to X$ is the required morphism. $\square$

Lemma 3.13. *Let $(X, S + B) \geq 0$ be a pair, where X is a normal affine variety, $S + B \geq 0$ is a $\mathbb{Q}$-divisor, $K_X + S + B$ is $\mathbb{Q}$-Cartier, and $\lfloor S + B \rfloor = S$ is reduced and irreducible. Also assume that (S^n, B_{S^n}) is KLT, where $S^n \to S$ is the normalization morphism and $(K_X + S + B)|_{S^n} = K_{S^n} + B_{S^n}$, and $f : Y \to X$ as in Proposition 3.12. Then for every Weil divisor D in Y, there exists an open set $W \subseteq Y$ (depending on D) containing all codimension 1 points of S' such that $D|_W$ is $\mathbb{Q}$-Cartier.*

Proof. From the construction of $f : Y \to X$, we see that it is an isomorphism at the points where S is normal. By Corollary 3.9, S is normal in codimension 1. Therefore, by Proposition 3.11, Y is $\mathbb{Q}$-factorial at every codimension 1 point of S' and the required open set W exists. $\square$

Lemma 3.14. *Let $(X, S+B)$ be a pair, where $X = \operatorname{Spec} R$ is a normal variety, $S+B \geq 0$ is a $\mathbb{Q}$-divisor, $K_X + S + B$ is $\mathbb{Q}$-Cartier, and $\lfloor S+B \rfloor = S$ is reduced and irreducible. Let (S^n, B_{S^n}) be KLT, where $S^n \to S$ is the normalization morphism and $(K_X+S+B)|_{S^n} = K_{S^n} + B_{S^n}$ is defined by adjunction. Assume further that $f : Y \to X$ is a projective birational morphism from a normal variety Y and S' is the strict transform of S such that $f|_{S'} : S' \to S$ is the normalization morphism (such f exists by Proposition 3.12), and*

$$K_Y + S' = f^*(K_X + S + B) + \boldsymbol{A}_Y.$$

Then $\lceil \boldsymbol{A}_Y \rceil|_{S'}$ is an effective $\mathbb{Q}$-divisor on S'.

Proof. First observe that the restriction of $\lceil \mathbf{A}_Y \rceil$ to S' is well-defined by Lemma 3.13. If $\lceil \mathbf{A}_Y \rceil|_{S'}$ not effective, then there exists an exceptional divisor E_i in $\mathbf{A}_Y$ with coefficient $r_i \leq -1$ such that $\operatorname{codim}_{S'}(E_i \cap S') = 1$. Let $\mathfrak{p} \in X$ be the image of the generic point of an irreducible component of $E_i|_{S'}$ under the map f. The height of $\mathfrak{p}$ in R is 2, since $f|_{S'} : S' \to S$ is the normalization morphism by Proposition 3.12. Let $X_{\mathfrak{p}} = \operatorname{Spec} R_{\mathfrak{p}}$ and $Y_{\mathfrak{p}} = X_{\mathfrak{p}} \times_X Y$. Then $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are both excellent surfaces. Choose a log resolution $g : Z \to Y_{\mathfrak{p}}$ of $(Y_{\mathfrak{p}}, S'_{\mathfrak{p}} - \mathbf{A}_{Y_{\mathfrak{p}}})$. Then g induces a log resolution of $(X_{\mathfrak{p}}, S_{\mathfrak{p}} + B_{\mathfrak{p}})$ as well. Since $(S_{\mathfrak{p}}, B_{S_{\mathfrak{p}}}) = (S^n_{\mathfrak{p}}, B_{S^n_{\mathfrak{p}}})$ is KLT, by the connectedness lemma (Lemma 3.8), we get a contradiction. $\square$

Proposition 3.15. *Let $(X, S+B)$ be a pair, where $X = \operatorname{Spec} R$ is a normal variety, $S+B \geq 0$ is a $\mathbb{Q}$-divisor, K_X+S+B is $\mathbb{Q}$-Cartier, and $\lfloor S+B \rfloor = S$ is reduced and irreducible. Also assume that $f : Y \to X$ is a projective birational morphism from a normal variety Y and S' is the strict transform of S such that $f|_{S'} : S' \to S$ is the normalization morphism (such f exists by Proposition 3.12), and*

$$K_Y + S' = f^*(K_X + S + B) + \boldsymbol{A}_Y. \tag{3.4}$$

Then there exists a $\mathbb{Q}$-divisor $\Xi \geq 0$ on Y satisfying the following properties:
(i) $\Xi \geq S' + \{-\boldsymbol{A}_Y\}$ and $\lfloor \Xi \rfloor = S'$,

(ii) $(p^e - 1)(K_Y + \Xi)$ *is an integral Weil divisor for some* $e > 0$*, and*

(iii) $\lceil \boldsymbol{A}_Y \rceil - (K_Y + \Xi)$ *is* f*-ample.*

Proof. To construct such a divisor Ξ, we first construct an effective Cartier divisor F on Y such that $-F$ is f-ample. Since X is affine and f is birational, there exists an f-ample divisor $\mathcal{A}$ and an effective Cartier divisor F (not necessarily exceptional) not containing the support of S' such that $\mathcal{A} + F \sim 0$, i.e., $-F$ is f-ample and $S' \nsubseteq \mathrm{Supp}F$.

Now rewrite the equation (3.4) in the following way

$$\lceil \mathbf{A}_Y \rceil - (K_Y + S' + \{-\mathbf{A}_Y\} + \varepsilon F) \sim_{\mathbb{Q}} -f^*(K_X + S + B) - \varepsilon F \tag{3.5}$$

where $\varepsilon > 0$.

Notice that both sides of the above relation (3.5) are $\mathbb{Q}$-Cartier divisors. Let G be the reduced divisor of codimension 1 components of the exceptional locus of f and H a sufficiently ample divisor on Y such that $\mathcal{O}_Y(H - \lceil \mathbf{A}_Y \rceil - G)$ and $\mathcal{O}_Y(K_Y + H)$ are both globally generated. Let $D \geq 0$ be a divisor whose support does not contain S' but $D \sim H - \lceil \mathbf{A}_Y \rceil - G$, then $D + G \sim H - \lceil \mathbf{A}_Y \rceil$. By (3.5), $K_Y + \Xi'$ is $\mathbb{Q}$-Cartier where

$$\Xi' = S' + \{-\mathbf{A}_Y\} + D + \varepsilon F + G \sim S' + \{-\mathbf{A}_Y\} + \varepsilon F + H - \lceil \mathbf{A}_Y \rceil.$$

Since $\mathcal{O}_Y(K_Y + H)$ is globally generated, there exists a divisor $E \geq 0$ whose support does not contain S' such that $E - K_Y \sim H$ is Cartier. Let $\Delta = \frac{1}{p^{e_0}-1}(\Xi' + E)$, where $e_0 \gg 0$, then $\Delta \sim_{\mathbb{Q}} \frac{1}{p^{e_0}-1}((K_Y + \Xi') + H)$, so $\Delta \geq 0$ is $\mathbb{Q}$-Cartier. Thus $K_Y + \Xi''$ is $\mathbb{Q}$-Cartier and $p \nmid \mathrm{index}(K_Y + \Xi'')$, where $\Xi'' = \Xi' + \Delta = S' + \{-\mathbf{A}_Y\} + \varepsilon F + D + G + \Delta$. We replace the S' contained in Δ by an integral Weil divisor $S_1 \geq 0$ such that $S' \sim S_1$ and S_1 does not contain S', then we still have $K_Y + \Xi''$ is $\mathbb{Q}$-Cartier and $p \nmid \mathrm{index}(K_Y + \Xi'')$.

We can rewrite the relation (3.5) in the following way

$$\lceil \mathbf{A}_Y \rceil - (K_Y + \Xi'' - D - G) \sim_{\mathbb{Q}} -f^*(K_X + S + B) - \varepsilon F - \Delta. \tag{3.6}$$

Let $\Xi = \Xi'' - D - G$. Then from the relation above, we get that $\lceil \mathbf{A}_Y \rceil - (K_Y + \Xi)$ is a $\mathbb{Q}$-Cartier f-ample divisor for $e_0 \gg 0$, since $-F$ is f-ample and the coefficients of Δ are small for $e_0 \gg 0$. Also notice that the denominators of $K_Y + \Xi$ are still not divisible by p. Thus Ξ satisfies all the three properties stated above. $\square$

Lemma 3.16. *With the same notations and hypothesis as in the Proposition 3.15, further assume that (S^n, B_{S^n}) is strongly F-regular, where $S^n \to S$ is the normalization morphism and $(K_X + S + B)|_{S^n} = K_{S^n} + B_{S^n}$ is defined by adjunction. Then we can choose the divisor Ξ to satisfy additionally the following properties*
(iv) $\Xi \leq S' + \{-\mathbf{A}_Y\} + f^*A$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $A \geq 0$ on X and
(v) $(S^n, B^*_{S^n})$ is strongly F-regular, where $B^*_{S^n} = B_{S^n} + A|_{S^n}$.

Proof. Let A be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X whose support contains $f(\mathrm{Ex}(f))$, $\mathrm{Supp}(B)$ and also $f(D), f(H), f(E), f(F)$, and $f(S_1)$ which appeared during the construction of Ξ in Proposition 3.15, but not the $\mathrm{Supp}(S)$, such that $(S^n, B_{S^n} + A|_{S^n})$ is strongly F-regular.

Recall from the proof of Proposition 3.15 that $\Xi = \Xi'' - D - G = S' + \{-\mathbf{A}_Y\} + \varepsilon F + \frac{1}{p^{e_0}-1}(\Xi' + E)$. Thus by choosing $e_0 \gg 0$ and $0 < \varepsilon \ll 1$, we can guarantee that Ξ satisfies both of the properties (iv) and (v). $\square$

3.2 Main Theorem

In this section, we a prove the main theorem of this chapter which is the characteristic $p > 0$ analogue of PLT inversion of adjunction.

Theorem 3.17 (Inversion of Adjunction)**.** *Let $(X, S+B)$ be a pair, where X is a normal variety, $S + B \geq 0$ is a $\mathbb{Q}$-divisor, $K_X + S + B$ is $\mathbb{Q}$-Cartier, and $S = \lfloor S + B \rfloor$ is reduced and irreducible. Let $\nu : S^n \to S$ be the normalization morphism, write $(K_X + S + B)|_{S^n} = K_{S^n} + B_{S^n}$. If (S^n, B_{S^n}) is strongly F-regular, then S is normal; furthermore, S is a unique center of sharp F-purity of $(X, S+B)$ in a neighborhood of S and $(X, S+B)$ is purely F-regular near S.*

Proof. **Normality of** S: Since the question is local on the base, we can assume that X is an affine variety. Let $f : Y \to X$ be a projective birational morphism from a normal variety Y and S' the strict transform of S such that $f|_{S'} : S' \to S$ is the normalization morphism (such f exists by Proposition 3.12), and

$$K_Y + S' = f^*(K_X + S + B) + \mathbf{A}_Y. \tag{3.7}$$

Claim: The image of the map

$$f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) \xrightarrow{\delta} (f|_{S'})_*\mathcal{O}_{S'}(\lceil \mathbf{A}_Y \rceil|_{S'}) \tag{3.8}$$

contains $\nu_*\mathcal{O}_{S^n}$.

Grant (3.8) for the time being, then since

$$f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) \subseteq \mathcal{O}_X$$

(as $\lceil \mathbf{A}_Y \rceil$ is exceptional), it follows that the morphism $\mathcal{O}_X \to \nu_*\mathcal{O}_{S^n}$ is surjective. This implies that $\nu_*\mathcal{O}_{S^n} = \mathcal{O}_S$, hence $S = S^n$.

Proof of Claim (3.8): We have the following short exact sequence

$$0 \to \mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil - S') \to \mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) \to Q \to 0 \tag{3.9}$$

where $Q \to \mathcal{O}_{S'}(\lceil \mathbf{A}_Y \rceil|_{S'})$ is the natural map and $\lceil \mathbf{A}_Y \rceil|_{S'}$ is well defined by Lemma 3.13.

Let $\Xi \geq 0$ be a $\mathbb{Q}$-divisor on Y as in the conclusion of the Proposition 3.15 and 3.16 and $\Xi'' \geq 0$ is another $\mathbb{Q}$-divisor on Y which appeared in the proof of the Proposition 3.15. Let $g > 0$ be an integer such that $(p^g - 1)(K_Y + \Xi'')$ is a Cartier divisor and $(p^g - 1)(K_Y + \Xi)$ is an integral Weil divisor. Such integer $g > 0$ exists by the definition of Ξ'' and Property (ii) of Ξ in Proposition 3.15. Also assume that $L_{eg,\Xi} = (1 - p^{eg})(K_Y + \Xi)$. Then from (3.6), we have

$$(p^{eg} - 1)\lceil \mathbf{A}_Y \rceil + L_{eg,\Xi} = (p^{eg} - 1)(\lceil \mathbf{A}_Y \rceil - (K_Y + \Xi)) \sim (p^{eg} - 1)(H - (K_Y + \Xi''))$$

is an ample Cartier divisor. Twisting the exact sequence (3.9) by the ample line bundle $\mathcal{O}_Y((p^{eg} - 1)\lceil \mathbf{A}_Y \rceil + L_{eg,\Xi})$ and taking cohomologies, we get the following diagram

$$\begin{tikzcd}
0 \arrow[r] & F^{eg}_* f_* L \arrow[r] \arrow[d] & F^{eg}_* f_* M \arrow[r, "\gamma_e"] \arrow[d, "\alpha_e"'] & F^{eg}_*(f|_{S'})_* N \arrow[r] \arrow[d, "\beta_e"] & 0 \\
0 \arrow[r] & f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil - S') \arrow[r] & f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) \arrow[r, "\delta"] & (f|_{S'})_*\mathcal{O}_{S'}(\lceil \mathbf{A}_Y \rceil|_{S'}) &
\end{tikzcd} \tag{3.10}$$

where $L = \mathcal{O}_Y(L_{eg,\Xi} + p^{eg}\lceil \mathbf{A}_Y \rceil - S')$, $M = \mathcal{O}_Y(L_{eg,\Xi} + p^{eg}\lceil \mathbf{A}_Y \rceil)$ and $N = Q \otimes \mathcal{O}_Y((p^{ge} - 1)\lceil \mathbf{A}_Y \rceil + L_{eg,\Xi})$. The top sequence is exact since

$$\begin{aligned} & R^1 f_* \mathcal{O}_Y(L_{eg,\Xi} + p^{eg}\lceil \mathbf{A}_Y \rceil - S') \\ & = R^1 f_* \mathcal{O}_Y((\lceil \mathbf{A}_Y \rceil - S') + (p^{eg} - 1)(\lceil \mathbf{A}_Y \rceil - K_Y - \Xi)) \\ & = 0 \text{ for } e \gg 0, \end{aligned}$$

by Property (iii) of Ξ in the Proposition 3.15 and the Serre Vanishing theorem. Existence of the vertical morphisms in the digram (3.10) is guaranteed by Lemma 3.18. From the commutativity of the above diagram (3.10), we get that

$$\operatorname{Image}(\alpha_e) \stackrel{\delta}{\twoheadrightarrow} \operatorname{Image}(\beta_e) \tag{3.11}$$

is surjective, for all $e \gg 0$.

Also we have the following commutative diagram

$$\begin{tikzcd} F^{eg}_*(f|_{S'})_*(Q \otimes \mathcal{O}_Y(L_{eg,\Xi} + (p^{eg}-1)\lceil \mathbf{A}_Y \rceil)) \arrow[r] \arrow[d, "\beta_e"] & F^{eg}_*(f|_{S'})_*\mathcal{O}_{S'}((L_{eg,\Xi})|_{S'} + p^{eg}\lceil \mathbf{A}_Y \rceil|_{S'}) \arrow[d, "\psi_e"] \\ (f|_{S'})_*\mathcal{O}_{S'}(\lceil \mathbf{A}_Y \rceil|_{S'})) \arrow[r, equal] & (f|_{S'})_*\mathcal{O}_{S'}(\lceil \mathbf{A}_Y \rceil|_{S'}) \end{tikzcd} \tag{3.12}$$

where $(L_{eg,\Xi})|_{S'} = (1 - p^{eg})(K_{S'} + \Xi_{S'})$, $\Xi_{S'}$ is an effective $\mathbb{Q}$-divisor on S' defined by adjunction such that $K_{S'} + \Xi_{S'} = (K_Y + \Xi)|_{S'}$. Observe that the adjunction formula makes sense because $\mathcal{O}_Y(m(K_Y + \Xi))$ is locally free at all condimension 1 points of S' for some $m > 0$ by Lemma 3.13 and so all the hypothesis of [Kol13, Definition 4.2] are satisfied.

Clearly $\operatorname{Image}(\beta_e) \hookrightarrow \operatorname{Image}(\psi_e)$. We will prove that $\operatorname{Image}(\beta_e)$ contains $\nu_*\mathcal{O}_{S^n}$ for all $e \gg 0$.

Since $\{-\mathbf{A}_Y\} - \lceil \mathbf{A}_Y \rceil = -\mathbf{A}_Y$, the inequality (iv) in Proposition 3.16 implies (after adding $K_Y - \lceil \mathbf{A}_Y \rceil$ and restricting to S') that

$$h^*(K_{S^n} + B_{S^n} + A|_{S^n}) \geq K_{S'} + \Xi_{S'} - \lceil \mathbf{A}_Y \rceil|_{S'} \tag{3.13}$$

where $h : S' \to S^n$ is the induced morphism.
Since $\lceil \mathbf{A}_Y \rceil|_{S'}$ is effective by Lemma 3.14, from (3.13) we get

$$(1 - p^e)h^*(K_{S^n} + B^*_{S^n}) \leq (1 - p^e)(K_{S'} + \Xi_{S'}) + p^e\lceil \mathbf{A}_Y \rceil|_{S'}$$

and so

$$F^e_*\mathcal{O}_{S^n}((1-p^e)(K_{S^n}+B^*_{S^n}))\subseteq h_*F^e_*\mathcal{O}_{S'}((1-p^e)(K_{S'}+\Xi_{S'})+p^e\lceil\mathbf{A}_Y\rceil|_{S'}). \quad (3.14)$$

Since $(S^n, B^*_{S^n})$ is strongly F-regular, by perturbing $B^*_{S^n}$ a little bit, we can assume that $p\nmid$ index$(K_{S^n}+B^*_{S^n})$ and $(S^n, B^*_{S^n})$ is still strongly F-regular (see [HX13, 2.13]). Let $\widehat{Q}=Q/\text{torsion}$. Observe that $\widehat{Q}$ is a rank 1 torsion free sheaf on S' and $\widehat{Q}\hookrightarrow\mathcal{O}_{S'}(\lceil\mathbf{A}_Y\rceil|_{S'})$. Let C be an effective Cartier divisor on S^n containing $h(\mathrm{Supp}(\mathcal{O}_{S'}(\lceil\mathbf{A}_Y\rceil|_{S'})/\widehat{Q}))$. Then $(S^n, B^*_{S^n}+\epsilon' C)$ is strongly F-regular for $0<\epsilon'\ll 1$. For $e\gg 0$ (where e depends on $\epsilon'>0$), we get the following factorizations of morphisms

$$\mathcal{O}_{S'}(\lceil\mathbf{A}_Y\rceil|_{S'}-(p^e-1)h^*(\epsilon' C))\hookrightarrow\widehat{Q}\hookrightarrow\mathcal{O}_{S'}(\lceil\mathbf{A}_Y\rceil|_{S'}). \quad (3.15)$$

Combining all these, we get the following commutative diagram

$$\begin{array}{ccc}
\nu_*F^{eg'}_*\mathcal{O}_{S^n}((1-p^{eg'})(K_{S^n}+B^*_{S^n}+\epsilon' C)) & \twoheadrightarrow & \nu_*\mathcal{O}_{S^n} \\
\Big\downarrow{\scriptstyle cf.\ (3.14)} & & \\
\nu_*h_*F^{eg'}_*\mathcal{O}_{S'}\Big((p^{eg'}-1)(\lceil\mathbf{A}_Y\rceil|_{S'}-K_{S'}-\Xi_{S'})+\Big(\lceil\mathbf{A}_Y\rceil|_{S'}-(p^{eg'}-1)\epsilon' h^*C\Big)\Big) & & \\
\Big\downarrow{\scriptstyle (3.15)} & & \Big\downarrow \\
\nu_*h_*F^{eg'}_*\Big(\mathcal{O}_{S'}\Big((p^{eg'}-1)(\lceil\mathbf{A}_Y\rceil|_{S'}-K_{S'}-\Xi_{S'})\Big)\otimes\widehat{Q}\Big) & & \\
\Big\downarrow{\scriptstyle (3.15)} & & \\
\nu_*h_*F^{eg'}_*\mathcal{O}_{S'}\Big((1-p^{eg'})(K_{S'}+\Xi_{S'})+p^{eg'}\lceil\mathbf{A}_Y\rceil|_{S'}\Big) & \longrightarrow & \nu_*h_*\mathcal{O}_{S'}(\lceil\mathbf{A}_Y\rceil|_{S'})
\end{array} \quad (3.16)$$

Since $(S^n, B^*_{S^n}+\epsilon' C)$ is strongly F-regular and ν is a finite morphism, the top horizontal row of the diagram (3.16) is surjective for all $e\gg 0$. This implies that the image, Image(β_e) of the map

$$\nu_*h_*F^{eg'}_*(\mathcal{O}_{S'}((p^{eg'}-1)(\lceil\mathbf{A}_Y\rceil|_{S'}-K_{S'}-\Xi_{S'}))\otimes Q)\xrightarrow{\beta_e}\nu_*h_*\mathcal{O}_{S'}(\lceil\mathbf{A}_Y\rceil|_{S'})$$

contains $\nu_*\mathcal{O}_{S^n}$ for all $e\gg 0$, since β_e factors through

$$\nu_*h_*F^{eg'}_*(\mathcal{O}_{S'}((p^{eg'}-1)(\lceil\mathbf{A}_Y\rceil|_{S'}-K_{S'}-\Xi_{S'}))\otimes\widehat{Q}).$$

Combining this with (3.11), we get our Claim (3.8).

Uniqueness of the F-pure Center: Now we will prove that S is the unique center of sharp F-purity of $(X, S+B)$ in a neighborhood of S. Recall that

$$\mathrm{Image}(\alpha_e) \subseteq f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) \subseteq \mathcal{O}_X.$$

Since S normal, from the proof of the claim (3.8) we get that $\mathrm{Image}(\alpha_e)$ surjects onto $\mathcal{O}_S$, hence $\mathrm{Image}(\alpha_e) = f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) = \mathcal{O}_X$ near S, i.e., $\lceil \mathbf{A}_Y \rceil$ is effective and exceptional over a neighborhood of S.

Now, if possible, let Z be a center of sharp F-purity of $(X, S+B)$ such that $Z \cap S \neq \emptyset$. Let D_1 be an effective $\mathbb{Q}$-Cartier divisor on X such that $S \nsubseteq D_1$ and $p \nmid \mathrm{index}(K_X + S + B + D_1)$. Such D_1 exists by [HX13, 2.13] since X is affine and hence $S \sim T$, where $T \geq 0$ and $\mathrm{Supp}\, T \not\supseteq S$. Choose D_2, a Cartier divisor such that $Z \subseteq D_2$ but $S \nsubseteq D_2$. Choose the coefficients of D_1 sufficiently small and $1 \gg \delta > 0$, so that $\Xi + f^*(D)$ satisfies all the properties of Ξ, where $D = D_1 + \delta D_2$. Then running through the same proof as above with Ξ replaced by $\Xi + f^*(D)$, we get that $\mathrm{Image}(\alpha_e) = f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) = \mathcal{O}_X$ near S, i.e.,

$$F^{eg}_* f_*\mathcal{O}_Y\left((1-p^{eg})(K_Y + \Xi + f^*(D) + p^{eg}\lceil \mathbf{A}_Y \rceil\right) \to f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) = \mathcal{O}_X \tag{3.17}$$

is surjective near S for all $e \gg 0$.

Now

$$\begin{aligned}
&((1-p^{eg})(K_Y + \Xi + f^*(D)) + p^{eg}\lceil \mathbf{A}_Y \rceil) - ((1-p^{eg}) f^*(K_X + S + B + D)) \\
&= \lceil \mathbf{A}_Y \rceil - (p^{eg} - 1)(\Delta + \epsilon F) \\
&= \lceil \mathbf{A}_Y \rceil - \frac{p^{eg}-1}{p^{e_0}-1}(\Xi' + E) - (p^{eg}-1)\epsilon F \leq 0
\end{aligned}$$

for sufficiently large and divisible $e > 0$, since $\mathrm{Supp} \lceil \mathbf{A}_Y \rceil \subseteq \mathrm{Supp}\, G \subseteq \mathrm{Supp}\, \Xi'$, where G is the reduced divisor of codimension 1 components of the exceptional locus of f.

This gives the following commutative diagram near S

$$\begin{tikzcd}
0 \arrow[r] & F^{eg}_* f_*\mathcal{O}_Y((1-p^{eg})(K_Y + \Xi + f^*(D)) + p^{eg}\lceil \mathbf{A}_Y \rceil) \arrow[r] \arrow[d] & F^{eg}_*\mathcal{O}_X((1-p^{eg})(K_X + S + B + D)) \arrow[d] \\
0 \arrow[r] & f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) \arrow[r, equal] & \mathcal{O}_X
\end{tikzcd}$$

Since the image of the second vertical map stabilizes to $\sigma(X, S+B+D)$ for $e \gg 0$, by (3.17), we see that $\sigma(X, S+B+D) = \mathcal{O}_X$ near S. Thus $(X, S+B+D)$ is sharply F-pure near S. Hence Z is not a center of F-purity for $(X, S+B)$, a contradiction.

F-**regular Inversion of Adjunction**: For any effective Cartier divisor E not containing S, in the proof above, we may assume that $\operatorname{Supp} E \subseteq \operatorname{Supp} D$ and hence the natural map $\mathcal{O}_X \to F_*^{eg}\mathcal{O}_X(\lceil (p^{eg}-1)(S+B)\rceil + E)$ splits near S. Therefore, $(X, S+B)$ is purely F-regular near S.

□

Lemma 3.18. *The vertical morphisms in the diagram* (3.10) *are well defined.*

Proof. First $\alpha_e : F_*^{eg} f_*\mathcal{O}_Y(L_{eg,\Xi} + p^e\lceil \mathbf{A}_Y\rceil) \to f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y\rceil)$ is defined naturally by the Grothendieck trace map (see [BS13]) followed by the twist of $\lceil \mathbf{A}_Y\rceil$ (see Proposition 2.20) and f_*.

$$\begin{tikzcd}
0 \arrow[r] & F_*^{eg} f_* L \arrow[r] \arrow[d] & F_*^{eg} f_* M \arrow[r, "\gamma_e"] \arrow[d, "\alpha_e"'] & F_*^{eg}(f|_{S'})_* N \arrow[r] \arrow[d, "\beta_e"] & 0 \\
0 \arrow[r] & f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y\rceil - S') \arrow[r] & f_*\mathcal{O}_Y(\lceil \mathbf{A}_Y\rceil) \arrow[r, "\delta"] & (f|_{S'})_*\mathcal{O}_{S'}(\lceil \mathbf{A}_Y\rceil|_{S'}) &
\end{tikzcd}$$

To define the first vertical map, we need to do some work. Let U be the smooth locus of Y. Since $\lfloor \Xi \rfloor = S'$ is irreducible and Y is normal, $S'|_U$ is a center of F-purity of $(U, \Xi|_U)$. Then by [Sch14, 5.1], there exists a map (following the Grothendieck trace map)

$$F_*^{eg}\mathcal{O}_U((L_{eg,\Xi} - S')|_U) \to \mathcal{O}_U(-S'|_U).$$

Twisting this map by $\lceil \mathbf{A}_Y\rceil|_U$, we get

$$F_*^{eg}\mathcal{O}_U((L_{eg,\Xi} + p^{eg}\lceil \mathbf{A}_Y\rceil - S')|_U) \to \mathcal{O}_U((\lceil \mathbf{A}_Y\rceil - S')|_U).$$

Since $\operatorname{codim}_Y(Y-U) \geq 2$, this map extends (uniquely) to a map on Y:

$$F_*^{eg}\mathcal{O}_Y(L_{eg,\Xi} + p^{eg}\lceil \mathbf{A}_Y\rceil - S') \to \mathcal{O}_Y(\lceil \mathbf{A}_Y\rceil - S')$$

as all of the sheaves considered above are reflexive. Applying f_* to this map we get our first vertical map.

We define β_e by diagram chasing. It is easy to see that β_e is well defined. □

Corollary 3.19. *With the same hypothesis as Theorem* 3.17, $(X, S+B)$ *is PLT near* S*.*

Proof. Since $(X, S + B)$ is purely F-regular near S by Theorem 3.17, it is PLT near S by [HW02, 3.3]. $\square$

CHAPTER 4

VANISHING THEOREMS AND LOG CANONICAL CENTERS

The Kawamata-Viehweg vanishing theorem, which is a generalization of the Kodaira vanishing theorem, is one of the fundamental tools used in the study of the Minimal Model Program in characteristic 0. Unfortunately, this theorem fails in characteristic $p > 0$. In this chapter, we prove a special version of this theorem in characteristic $p > 5$ for 3-folds, using the existence of minimal models for 3-folds due to Hcaon and Xu [HX13], and Birkar [Bir13]. We then use this theorem to prove the normality of minimal LC centers.

The following theorem is one of the common versions of the Kawamata-Viehweg vanishing theorem used in characteristic 0.

Theorem 4.1. *Let $f : X \to Y$ be a projective morphism between two quasi-projective varieties in characteristic 0. Let (X, Δ) be a KLT pair and D, an integral Weil divisor on X such that $D \equiv \Delta + M$, where M is f-nef and f-big. Then $R^i f_* \mathcal{O}_X(K_X + D) = 0$ for all $i > 0$.*

For the rest of this chapter we work over an algebraically closed field k of characteristic $p > 5$, unless stated otherwise.

Acknowledgement: Contents of this chapter are from the article [DH15].

4.1 Properties of Log Canonical Centers

In this section, we establish some basic properties of the LC centers which will be useful later in this chapter.

Lemma 4.2. *Let X be a $\mathbb{Q}$-factorial KLT 3-fold and $(X, \Delta \geq 0)$, a log canonical pair. Let W_1 and W_2 be two log canonical centers of (X, Δ). Then every irreducible component of*

$W_1 \cap W_2$ is a log canonical center of (X, Δ).

Proof. There are three cases depending on the codimension of W_1 and W_2.

Case I: *The codimensions of W_1 and W_2 are both* 1. In this case, W_1 and W_2 are components of Δ. Let $\Delta = W_1 + W_2 + \bar{\Delta}$. Then by adjunction on W_1^n, we have $(K_X + W_1 + W_2 + \bar{\Delta})|_{W_1^n} = K_{W_1^n} + W_2|_{W_1^n} + \operatorname{Diff}_{W_1^n}(\bar{\Delta})$, where $W_1^n \to W_1$ is the normalization. By localizing at the generic point of an irreducible component of $W_1 \cap W_2$, we reduce it to a surface problem. Now, on a surface in characteristic $p > 0$, the relative Kwamata-Viehweg vanishing and Kollar's connectedness theorem hold (see [Kol13, 10.13] and [Das13, 3.1]). Thus on a surface, the intersection of two LC centers is a LC center and we are done.

Case II: *The codimension of W_1 is* 1 *and the codimension W_2 is* 2. Since X is $\mathbb{Q}$-facorial, $(X, (1-\epsilon)\Delta)$ is KLT for any $0 < \epsilon < 1$. Thus by [Bir13], there exists a $\mathbb{Q}$-factorial model $f' : X' \to (X, \Delta)$ of relative Picard number $\rho(X'/X) = 1$ such that $\operatorname{Ex}(f')$ is a unique exceptional divisor E' over W_2 and

$$K_{X'} + E' + W_1' + \Delta' = f'^*(K_X + \Delta) \tag{4.1}$$

where $\Delta' \geq 0$, and W_1' is the strict transform of W_1 under f'.

Since W_1' and E' are $\mathbb{Q}$-Cartier, they intersect along a curve (possible reducible). Let C' be an irreducible component of $W_1' \cap E'$. Then by Case I, C' is a LC center of $(X', E' + W_1' + \Delta' \geq 0)$. Since every irreducible component of $W_1 \cap W_2$ is dominated by an irreducible component of $W_1' \cap E'$, we are done by relation (4.1).

Case III: *The codimension of W_1 and W_2 are both* 2. Again since X is $\mathbb{Q}$-factorial, $(X, (1-\epsilon)D)$ is KLT for any $0 < \epsilon < 1$. Thus by [Bir13], there exists a $\mathbb{Q}$-factorial model $f' : X' \to (X, \Delta)$ such that $\operatorname{Ex}(f') = E_1' \cup E_2'$, where $f'(E_1') = W_1$ and $f'(E_2') = W_2$, and

$$K_{X'} + E_1' + E_2' + \Delta' = f'^*(K_X + \Delta). \tag{4.2}$$

Since E_1' and E_2' are $\mathbb{Q}$-Cartier, they intersect along a curve (possibly reducible). Let C' be an irreducible component of $E_1' \cap E_2'$. Then by Case I, C' is a LC center of $(X', E_1' + E_2' + \Delta' \geq 0)$. Since every irreducible component of $W_1 \cap W_2$ is dominated by an irreducible component of $E_1' \cap E_2'$, we are done by relation (4.2). $\square$

The following proposition is a characteristic $p > 5$ version of Fujino's adjunction theorem for DLT pairs (see [Cor07, 3.9.2] and [Kol13, 4.16]) on a $\mathbb{Q}$-factorial 3-fold.

Proposition 4.3 (DLT Adjunction)**.** *Let $(X, \Delta \geq 0)$ be a $\mathbb{Q}$-factorial DLT 3-fold such that $\Delta = D_1 + D_2 + \cdots + D_r + B$ and $\lfloor \Delta \rfloor = D_1 + D_2 + \cdots + D_r$. Also assume that X has KLT singularities. Then the following hold*

1. *The s-codimensional log canonical centers of (X, Δ) are exactly the irreducible components of the various intersections $D_{i_1} \cap \cdots \cap D_{i_s}$ for some $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, r\}$.*
2. *Every irreducible component of $D_{i_1} \cap \cdots \cap D_{i_s}$ is normal and of pure codimension s.*
3. *Let W be a log canonical center of (X, Δ), then there exists an effective $\mathbb{Q}$-divisor $\Delta_W \geq 0$ on W such that $(K_X + \Delta)|_W \sim_{\mathbb{Q}} K_W + \Delta_W$ and (W, Δ_W) is DLT.*
4. *If $D_i \cap D_j = \emptyset$ for all $i \neq j$, then (X, Δ) is in fact PLT.*

Proof. The proof in [Kol13, Theorem 4.16] works in characteristic $p > 5$ with some modification. We write the whole proof here for the sake of completeness.
Let E be a divisor over X such that $a(E, X, \Delta) = -1$ and $Z = \text{center}_X E$. By localizing at the generic point of Z, we may assume that Z is a closed point of X. By the DLT assumption, X is smooth at Z and Δ is SNC. If $\dim_Z X = n$, then there exist SNC divisors $B_1, B_2, \ldots, B_n$ through Z such that $\Delta = \sum a_i B_i$ for some $0 \leq a_i \leq 1$ (we can ignore the components of Δ that do not pass through Z). Set $\Delta' = \sum B_i$. Then $a(E, X, \Delta') \geq -1$ by [Kol13, 2.1] and $a(E, X, \Delta) > a(E, X, \Delta') \geq -1$ by [KM98, 2.27] unless $a_i = 1$ for all i. Thus every B_i appears in Δ with coefficient 1. Hence the B_i's are some of the D_j's and Z is an irreducible component of the intersection of the corresponding D_j's. This proves one direction of (1).

Since X is $\mathbb{Q}$-factorial, $(X, D_i + (1-\epsilon)(D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_r) + B)$ is DLT. Then by the equivalence of Part (1) and (3) in [KM98, Proposition 5.51], $(X, D_i + (1-\epsilon)(D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_r) + B)$ is PLT. Thus (X, D_i) is also PLT and then by adjunction, (D_i, Diff_{D_i}) is KLT. Since Diff_{D_i} has standard coefficients, by [Har98] and [HX13, 3.1], $(D_i, \mathrm{Diff}_{D_i})$ is strongly F-regular in characteristic $p > 5$. Then by [HX13, 4.1] and [Das13, 4.1, 5.4], D_i is normal. This proves that every irreducible component of $\lfloor \Delta \rfloor$ is normal and hence (2) for $s = 1$.

Next we prove the following claim. After that, we establish (1), (2), and (3) by induction on s.

Claim: If D is an irreducible component of $\lfloor \Delta \rfloor$, then (D, Δ_D) is DLT, where $\Delta_D = \mathrm{Diff}_D(\Delta - D)$. By adjunction (D, Δ_D) is log canonical. Let Z be a LC center of (D, Δ_D). Then there exists a divisor E_D over D whose center is Z and $a(E_D, D, \Delta_D) = -1$, where $\Delta_D = \mathrm{Diff}_D(\Delta - D)$. Thus by [Kol13, 4.8], there exists a divisor E_X over X whose center is Z and $a(E_X, X, \Delta) = -1$. Since (X, Δ) is DLT, (X, Δ) is SNC at the generic point of Z. Thus $\mathrm{Diff}_D(\Delta - D) = (\Delta - D)|_D$ and hence $(D, \mathrm{Diff}_D(\Delta - D))$ is SNC at the generic point of Z. Therefore, $(D, \mathrm{Diff}_D(\Delta - D))$ is DLT.

Every irreducible component of $D_i \cap D_j$ is a LC center of (X, Δ) by Lemma 4.2. This proves (1) for $s = 2$. For $s > 2$, we use induction and the equality

$$D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_s} = (D_{i_1}|_{D_{i_s}}) \cap \cdots \cap (D_{i_{s-1}}|_{D_{i_s}}). \tag{4.3}$$

From the Claim 4.1, it follows that $(D_i, \mathrm{Diff}_{D_i}(\Delta - D_i))$ is DLT. Then D_i is a $\mathbb{Q}$-factorial surface by [FT12, 6.3]. Thus as before, we can show that each irreducible component of $D_j|_{D_i}$ is normal. This proves Part (2) for $s = 2$. For $s > 2$, we use induction and the equality (4.3).

Part (3) follows from the Claim and by induction on the codimension of W.

If $D_i \cap D_j = \emptyset$ for all $i \neq j$, then by the equivalence of Part (1) and (3) of [KM98, Proposition 5.51], (X, Δ) is PLT. □

Definition 4.4 (Divisorial Extraction)**.** Let $(X, \Delta \geq 0)$ be a $\mathbb{Q}$-factorial 3-fold log canonical pair with a unique exceptional divisor E over X of discrepancy $a(E, X, \Delta) = -1$. A divisorial extraction is a $\mathbb{Q}$-factorial PLT model $f : (Y, E + \Delta' \geq 0) \to (X, \Delta)$ of relative Picard number $\rho(Y/X) = 1$, such that $K_Y + E + \Delta' = f^*(K_X + \Delta)$.

Remark 4.5. Divisorial extractions exist in any dimension in characteristic 0 by [BCHM10] and [KK10, 3.1], and in dimension 3 and in characteristic $p > 5$ by [HX13] and [Bir13].

4.2 Vanishing Theorem

We prove the following vanishing theorem for 3-folds in characteristic $p > 5$.

Theorem 4.6 (Relative Vanishing Theorem)**.** *Let $(X, \Delta > 0)$ be a $\mathbb{Q}$-factorial 3-fold log canonical pair with isolated center W, $codim_X W = 2$ and S a unique exceptional divisor dominating W with $a(S, X, \Delta) = -1$. Also assume that X has KLT singularities. Let $f : (Y, S + B) \to (X, W)$ be the corresponding divisorial extraction such that $K_Y + S + B = f^*(K_X + \Delta)$. Then $R^1 f_* \mathcal{O}_Y(-S) = 0$.*

Proof. Note that $-S$ is $\mathbb{Q}$-Cartier f-ample divisor.

Claim: The following sequence is exact at all codimension 2 points of Y

$$0 \longrightarrow \mathcal{B}_e \longrightarrow F^e_* \mathcal{O}_Y((1-p^e)K_Y - p^e S) \xrightarrow{\phi_e} \mathcal{O}_Y(-S) \longrightarrow 0 \tag{4.4}$$

for all $e \gg 0$ and sufficiently divisible, where $\mathcal{B}_e$ is the kernel of ϕ_e.

Granting Claim (4.4) for the time being, we will show that $R^1 f_* \mathcal{O}_Y(-S) = 0$. The exact sequence (4.4) can be split into the following two exact sequences

$$0 \longrightarrow \mathcal{B}_e \longrightarrow F^e_* \mathcal{O}_Y((1-p^e)K_Y - p^e S) \xrightarrow{\phi_e} \mathrm{Im}(\phi_e) \longrightarrow 0 \tag{4.5}$$

and

$$0 \longrightarrow \mathrm{Im}(\phi_e) \longrightarrow \mathcal{O}_Y(-S) \longrightarrow \mathcal{Q}_e \longrightarrow 0 \tag{4.6}$$

where $\mathcal{Q}_e$ is the corresponding quotient.

Pushing forward the exact sequence (4.5) by f, we get

$$R^1 f_*(F^e_* \mathcal{O}_Y((1-p^e)K_Y - p^e S)) \to R^1 f_* \mathrm{Im}(\phi_e) \to R^2 f_* \mathcal{B}_e. \tag{4.7}$$

Now $R^2 f_* \mathcal{B}_e = 0$, since the maximum dimension of the fibers of f is 1. Let r be the index of $K_Y + S$ and $H = -(K_Y + S)$. By the division algorithm, there exist integers $k > 0$ and $0 \leq b < r$ such that $(p^e - 1) = r \cdot k + b$. Then by the Serre vanishing theorem

$$R^1 f_*(F^e_* \mathcal{O}_Y((1-p^e)K_Y - p^e S)) = F^e_*(R^1 f_* \mathcal{O}_Y(k \cdot rH - b(K_Y + S) - S)) = 0$$

for all $e \gg 0$ and sufficiently divisible, since H is f-ample.

Thus from (4.7), we get

$$R^1 f_* \mathrm{Im}(\phi_e) = 0. \tag{4.8}$$

Again, pushing forward the exact sequence (4.6), we get

$$R^1 f_* \mathrm{Im}(\phi_e) \to R^1 f_* \mathcal{O}_Y(-S) \to R^1 f_* \mathcal{Q}_e. \tag{4.9}$$

$R^1 f_* \mathcal{Q}_e = 0$, since $\mathcal{Q}_e$ is supported at finitely many points and $R^1 f_* \mathrm{Im}(\phi_e) = 0$ by (4.8). Thus we have

$$R^1 f_* \mathcal{O}_Y(-S) = 0. \tag{4.10}$$

We will now prove Claim (4.4). By Proposition 4.3, S is normal and $(Y, S + B)$ is PLT. Since Y is $\mathbb{Q}$-factorial, (Y, S) is also PLT.

Now, since the question is local on Y, we may assume that Y is affine. Then by [HX13, 2.13], we can choose an effective $\mathbb{Q}$-Cartier divisor $G \geq 0$ not containing S and with sufficiently small coefficients such that $K_Y + S + G$ is $\mathbb{Q}$-Cartier with index not divisible by p.

Localizing Y at a codimension 2 point, we may assume that Y is an excellent surface. Thus by adjunction, we have $(K_Y + S + G)|_S = K_S + D_S + G|_S$, where D_S is the Different. Since (Y, S) is PLT, (S, D_S) is KLT by adjunction. Hence (S, D_S) is strongly F-regular by [HX13, 2.2], since S is a smooth curve. Since the coefficients of G are sufficiently small, $(S, D_S + G|_S)$ is strongly F-regular. Therefore, we get the following surjection

$$F^e_* \mathcal{O}_S((1 - p^e)(K_S + D_S + G|_S)) \twoheadrightarrow \mathcal{O}_S$$

for all $e \gg 0$ and sufficiently divisible.

We have the following commutative diagram

$$\begin{tikzcd} F^e_* \mathcal{O}_Y((1 - p^e)(K_Y + S + G)) \arrow[r, two heads] \arrow[d] & F^e_* \mathcal{O}_S((1 - p^e)(K_S + D_S + G|_S)) \arrow[d, two heads] \\ \mathcal{O}_Y \arrow[r, two heads] & \mathcal{O}_S \end{tikzcd} \tag{4.11}$$

Since the ring $\mathcal{O}_Y$ is local, the surjectivity of the second vertical map (along with Nakayama's lemma) implies the surjectivity of the first vertical map, i.e.,

$$F^e_* \mathcal{O}_Y((1 - p^e)(K_Y + S + G)) \twoheadrightarrow \mathcal{O}_Y \qquad \text{is surjective.} \tag{4.12}$$

Since the map (4.12) factors through $F^e_* \mathcal{O}_Y((1 - p^e)K_Y)$, we get the following surjectivity

$$F^e_* \mathcal{O}_Y((1 - p^e)K_Y) \xrightarrow{\psi_e} \mathcal{O}_Y. \tag{4.13}$$

Let s be a pre-image of 1 under ψ_e. Then we get the following splitting of ψ_e

$$\mathcal{O}_Y \xrightarrow{\cdot s} F^e_*\mathcal{O}_Y((1-p^e)K_Y) \xrightarrow{\psi_e} \mathcal{O}_Y. \tag{4.14}$$

Twisting (4.14) by $\mathcal{O}_Y(-S)$ and taking reflexive hull, we get the following splitting

$$\mathcal{O}_Y(-S) \longrightarrow F^e_*\mathcal{O}_Y((1-p^e)K_Y - p^eS) \longrightarrow \mathcal{O}_Y(-S). \tag{4.15}$$

In particular, the morphism

$F^e_*\mathcal{O}_Y((1-p^e)K_Y - p^eS) \twoheadrightarrow \mathcal{O}_Y(-S)$ is surjective and Claim (4.4) follows.

$\square$

4.3 Minimal Log Canonical Centers

Normality of minimal log canonical centers is a consequence of the Kawamata-Viehweg vanishing theorem and inversion adjunction in characteristic 0. We take a similar approach here and use Theorem 4.6 in place of the Kawamata-Viehweg vanishing theorem.

Theorem 4.7. *Let (X, Δ) be a $\mathbb{Q}$-factorial 3-fold log canonical pair such that X has KLT singularities. If W is a minimal log canonical center of (X, Δ), then W is normal.*

Proof. Since X is $\mathbb{Q}$-factorial and all log canonical centers of (X, Δ) are contained in Δ, $(X, (1-\epsilon)\Delta)$ is KLT for any $0 < \epsilon < 1$. Then by Reid's Tie Breaking trick (see [Cor07, 8.7.1]), we may assume that W is the unique log canonical center of (X, Δ) with a unique exceptional divisor over X of discrepancy -1. There are two cases depending on the codimension of W.

Case I: *The codimension of W is* 1. Since X is $\mathbb{Q}$-factorial, (X, W) is log canonical. By adjunction $(K_X + W)|_{W^n} = K_{W^n} + \mathrm{Diff}_{W^n}$, where $W^n \to W$ is the normalization and $(W^n, \mathrm{Diff}_{W^n})$ is KLT. Thus by [Har98] and [HX13, 3.1], $(W^n, \mathrm{Diff}_{W^n})$ is strongly F-regular in characteristic $p > 5$. Then $W^n = W$, i.e., W is normal by [HX13, 4.1] or [Das13, 4.1].

Case II: *The codimension of W is* 2. Let $f : (Y, S+B) \to (X, \Delta)$ be a divisorial extraction such that

$$K_Y + S + B = f^*(K_X + \Delta)$$

where S is the exceptional divisor over W.

$(Y, S+B)$ is PLT with S an irreducible PLT center. Since Y is $\mathbb{Q}$-factorial, (Y, S) is also PLT. By adjunction, we have $(K_Y + S)|_{S^n} = K_{S^n} + \mathrm{Diff}_{S^n}$, where $S^n \to S$ is the normalization. Then $(S^n, \mathrm{Diff}_{S^n})$ is KLT. Hence by [HX13, 3.1], $(S^n, \mathrm{Diff}_{S^n})$ is strongly F-regular in characteristic $p > 5$, and so $S^n = S$, i.e., S is normal by [HX13, 4.1] or [Das13, 4.1, 5.4].

Consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-S) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_S \longrightarrow 0$$

By Theorem 4.6, we have $R^1 f_* \mathcal{O}_Y(-S) = 0$. Thus we get the following exact sequence

$$0 \longrightarrow f_*\mathcal{O}_Y(-S) \longrightarrow f_*\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_S \longrightarrow 0.$$

Since $f_*\mathcal{O}_Y(-S) = \mathscr{I}_W$ and $f_*\mathcal{O}_Y = \mathcal{O}_X$, we get

$$0 \longrightarrow \mathscr{I}_W \longrightarrow \mathcal{O}_X \longrightarrow f_*\mathcal{O}_S \longrightarrow 0.$$

Now $\mathcal{O}_X \twoheadrightarrow f_*\mathcal{O}_S$ factors in the following way

$$\begin{tikzcd}
& \mathcal{O}_W \arrow[dr, hook] & & \\
& & \nu_*\mathcal{O}_{W^n} \arrow[dr, hook] & \\
\mathcal{O}_X \arrow[ur, two heads] \arrow[rrr, two heads] & & & f_*\mathcal{O}_S
\end{tikzcd}$$

where $\nu : W^n \to W$ is the normalization morphism.
Hence $\mathcal{O}_W = \nu_*\mathcal{O}_{W^n}$, i.e., W is normal. $\square$

CHAPTER 5

ADJUNCTION FORMULA ON 3-FOLDS

Adjunction formula is an important technical tool in algebraic geometry. It is known for codimension 1 subvarieties in all dimension and in arbitrary characteristic. The simplest form of this formula is the following theorem.

Theorem 5.1. *[Har77, Proposition II.8.20] If X is a non-singular variety and $S \subseteq X$ is non-singular subvariety of codimension 1, then $(K_X + S)|_S \sim K_S$.*

If X is not smooth, then this formula requires a correction term called the 'Different'. We have the following general version of the formula.

Theorem 5.2. *Let $(X, S + \Delta)$ be a LC pair in arbitrary characteristic. Then there exists a $\mathbb{Q}$-divisor Δ_{S^n} on the normalization $S^n \to S$ called the 'Different' such that $(K_X + S + \Delta)|_{S^n} \sim_{\mathbb{Q}} K_{S^n} + \Delta_{S^n}$.*

Remark 5.3. For a more general version of Theorem 5.2 and some properties of the 'Different', see [Kol92, Chapter 6] and [Kol13, Section 4.1].

Adjunction formula for subvarieties of higher codimension is a conjecture proposed by Shokurov and Kawamata independently.

Conjecture 5.4. *[Amb99] Let $(X, \Delta \geq 0)$ be a pair, where X is a normal variety, $\Delta \geq 0$ is a $\mathbb{R}$-divisor, and $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $W \subseteq X$ be a subvariety of X which is a LC center of (X, Δ) and $W^n \to W$ is the normalization. Then*

1. *There exists an effective $\mathbb{R}$-divisor $\Delta_{W^n} \geq 0$ on W^n called the 'Different' such that $K_{W^n} + \Delta_{W^n}$ is $\mathbb{R}$-Cartier.*

2. *There exists a semi-ample $\mathbb{R}$-divisor $M_{W^n} \geq 0$ on W^n such that $(K_X + \Delta)|_{W^n} \sim_{\mathbb{R}} K_{W^n} + \Delta_{W^n} + M_{W^n}$.*

Many different (weaker) versions of this conjecture are known in characteristic 0. In 1997, Kawamata [Kaw97] proved the following theorem for LC center of condimension 2 in characteristic 0.

Theorem 5.5. *[Kaw97, Theorem 1] Let $(X, \Delta \geq 0)$ be LC pair, where X is a normal variety in characteristic 0 and $\Delta \geq 0$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Let W be a LC center of (X, Δ), $W^n \to W$ the normalization, and $\operatorname{codim}_X W = 2$. Then*

1. *There exists canonically determined effective $\mathbb{Q}$-divisors $M_W \geq 0$ and $\Delta_W \geq 0$ on W^n such that $(K_X + \Delta)|_{W^n} \sim_{\mathbb{Q}} K_W + \Delta_W + M_W$. Moreover, if $\Delta = \Delta' + \Delta''$ with Δ' (resp. Δ'') the sum of irreducible components which contain (resp. do not contain) W, then M_W is determined only by the pair (X, Δ').*

2. *There exists an effective $\mathbb{Q}$-divisor M'_W on W^n such that $M'_W \sim_{\mathbb{Q}} M_W$ and the pair $(W^n, \Delta_W + M'_W)$ is LC.*

3. *If X is KLT and W is a minimal LC center of (X, Δ), then there exists M'_W such that $(W, \Delta_W + M'_W)$ is KLT.*

In 1998, Kawamata [Kaw98] proved another version of the conjecture for LC center of arbitrary codimension in characteristic 0.

Theorem 5.6. *[Kaw98, Theorem 1] Let X be a normal projective variety in characteristic 0. Let $\Delta^0 \geq$ and $\Delta \geq 0$ be effective $\mathbb{Q}$-divisors on X such that $\Delta^0 < \Delta$, (X, Δ^0) is KLT, and (X, Δ) is LC. Let W be a minimal LC center of (X, Δ). Let H be an ample Cartier divisor on X, and $\epsilon > 0$ a positive rational number. Then there exists an effective $\mathbb{Q}$-divisor Δ_W on W such that*

$$(K_X + \Delta + \epsilon H)|_W \sim_{\mathbb{Q}} K_W + \Delta_W$$

and that the pair (W, Δ_W) is KLT. In particular, W has rational singularities.

In 2012, Fujino and Gongyo [FG12] proved the following special version of the conjecture in characteristic 0.

Theorem 5.7. *[FG12, Theorem 1.2] Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal projective variety in characteristic 0 and $\Delta \geq 0$ an* *effective* $\mathbb{K}$*-divisor on* X *such that* (X, Δ) *is LC. Let* W *be a minimal LC center of* (X, Δ)*. Then there exists an effective* $\mathbb{K}$*-divisor* $\Delta_W \geq 0$ *on* W *such that*

$$(K_X + \Delta)|_W \sim_{\mathbb{K}} K_W + \Delta_W$$

and the the pair (W, Δ_W) *is KLT. In particular,* W *has rational singularities.*

Remark 5.8. Observe that the theorem of Fujino-Gongyo (Theorem 5.7) is stronger than Kawamata's theorem (Theorem 5.6) in the sense that $\Delta = 0$ is allowed in Fujino-Gongyo's theorem. In the same article, Fujino and Gongyo also proved a local version of the adjunction formula; see [FG12, Theorem 7.2].

For the rest of this chapter, we work over an algebraically closed field of characteristic $p > 5$, unless stated otherwise.

Acknowledgement: Contents of this chapter are taken from the article [DH15].

5.1 Preliminaries

To start with, we will need the following definitions and results.

5.1.1 DCC Sets

We say that a set I of real numbers satisfies the *descending chain condition* or DCC, if it does not contain any infinite strictly decreasing sequence. For example,

$$I = \left\{ \frac{r-1}{r} : r \in \mathbb{N} \right\}$$

satisfies the DCC.

Let $I \subseteq [0, 1]$. We define

$$I_+ := \{j \in [0, 1] : j = \sum_{p=1}^{l} i_p \text{ for some } i_1, i_2, \ldots, i_l \in I\}$$

and

$$D(I) := \{a \leq 1 : a = \frac{m-1+f}{m}, m \in \mathbb{N}, f \in I_+\}.$$

Lemma 5.9. *[MP04, 4.4] Let* $I \subseteq [0, 1]$*. Then*

1. $D(D(I)) = D(I) \cup \{1\}$*.*

2. *I satisfies DCC if and only if $\bar{I}$ satisfies the DCC, where $\bar{I}$ is the closure of I.*

3. *I satisfies DCC if and only if $D(I)$ satisfies the DCC.*

Lemma 5.10. *[CGS14, Lemma 2.3][MP04, Lemma 4.3][HMX14, Lemma 4.1] Let (X, Δ) be a log canonical pair such that the coefficients of Δ belong to a set $I \subseteq [0,1]$. Let S be a normal irreducible component of $\lfloor \Delta \rfloor$ and $\Theta \geq 0$ be the $\mathbb{Q}$-divisor on S defined by adjunction:*

$$(K_X + \Delta)|_S = K_S + \Theta.$$

Then, the coefficients of Θ belong to $D(I)$.

5.1.2 Divisorial Parts and Moduli Parts

Let $f : X \to Z$ be a surjective proper morphism between two normal varieties and $K_X + D \sim_{\mathbb{Q}} f^*L$, where D is a $\mathbb{Q}$-divisor. Let (X, D) be LC near the generic fiber of f, i.e., $(f^{-1}U, D|_{f^{-1}U})$ is LC for some Zariski dense open subset $U \subseteq Z$. Then we define two divisors D_{div} and D_{mod} on Z in the following way:

$$\begin{aligned} D_{\text{div}} &= \textstyle\sum (1 - c_Q)Q, \text{ where } Q \subseteq Z \text{ are prime Weil divisors of } Z, \\ c_Q &= \sup\{c \in \mathbb{R} : (X, D + cf^*Q) \text{ is LC over the generic point } \eta_Q \text{ of } Q\} \text{ and} \\ D_{\text{mod}} &= L - K_Z - D_{\text{div}}, \text{ so that } K_X + D \sim_{\mathbb{Q}} f^*(K_Z + D_{\text{div}} + D_{\text{mod}}). \end{aligned}$$

5.1.3 Properties of D_{div} and D_{mod}

1. Observe that D_{div} is a fixed divisor on Z, called the *Divisorial part* and D_{mod} is a $\mathbb{Q}$-linear equivalence class on Z, called the *Moduli part*.

2. By abuse of language, f^*Q is defined as the divisor associated to the pullback f^*t of a local parameter t of the local ring $\mathcal{O}_{\eta_Q, Z}$. Since the supremum is defined over the generic point of Q, the choice of t is irrelevant.

3. If $f' : X' \to X$ is a proper birational morphism and D' is the log pullback of D, i.e.,

   $$K_{X'} + D' = f'^*(K_X + D),$$

   then $D'_{\text{div}} = D_{\text{div}}$. This happens because $(X', D' + (f \circ f')^*L)$ is LC if and only if $(X, D + f^*L)$ is LC, where L is a $\mathbb{Q}$-Cartier divisor on Z.

4. Let $f' : X' \to X$ be a finite morphism from a normal variety X' and D' the log pullback of D, i.e., $K_{X'} + D' = f'^*(K_X + D)$. If $\mathrm{char}(k) = 0$, or $\deg(f') < \mathrm{char}(k)$, or $\deg f' \nmid \mathrm{char}(k)$ and X'/X is Galois cover, then from [Kol13, Corollary 2.43], we see that (X', D') is LC if and only if (X, D) is LC. Thus $D'_{\mathrm{div}} = D_{\mathrm{div}}$.

5. If D is boundary over the generic point of every prime divisor $Q \subseteq Z$, then D_{div} is effective.

6. (X, D) is KLT (resp. LC) over the generic point of $Q \subseteq Z$ if and only if $c_Q > 0$ (resp. $c_Q \geq 0$).

7. If (X, D) is LC and D is boundary, then D_{div} is boundary.

Let $\overline{\mathcal{M}}_{0,n}$ be the moduli space of n-pointed stable curves of genus 0, $f_{0,n} : \overline{\mathcal{U}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ the universal family, and $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n$, the sections of $f_{0,n}$ which correspond to the marked points (see [Kee92] and [Knu83]). Let d_j $(j = 1, 2, \cdots, n)$ be rational numbers such that $0 < d_j \leq 1$ for all j, $\sum_j d_j = 2$ and $\mathcal{D} = \sum_j d_j \mathcal{P}_j$.

Lemma 5.11. *1. There exists a smooth projective variety $\mathcal{U}^*_{0,n}$, a $\mathbb{P}^1$-bundle $g_{0,n} : \mathcal{U}^*_{0,n} \to \overline{\mathcal{M}}_{0,n}$, and a sequence of blowups with smooth centers*

$$\overline{\mathcal{U}}_{0,n} = \mathcal{U}^{(1)} \xrightarrow{\sigma_2} \mathcal{U}^{(2)} \xrightarrow{\sigma_3} \cdots \xrightarrow{\sigma_{n-2}} \mathcal{U}^{(n-2)} = \mathcal{U}^*_{0,n}.$$

2. Let $\sigma : \overline{\mathcal{U}}_{0,n} \to \mathcal{U}^*_{0,n}$ and $g_{0,n} : \mathcal{U}^*_{0,n} \to \overline{\mathcal{M}}_{0,n}$ be the induced morphisms, and $\mathcal{D}^* = \sigma_* \mathcal{D}$. Then $K_{\overline{\mathcal{U}}_{0,n}} + \mathcal{D} - \sigma^*(K_{\mathcal{U}^*_{0,n}} + \mathcal{D}^*)$ is effective.

3. There exists a semi-ample $\mathbb{Q}$-divisor $\mathcal{L}$ on $\overline{\mathcal{M}}_{0,n}$ such that

$$K_{\mathcal{U}^*_{0,n}} + \mathcal{D}^* \sim_{\mathbb{Q}} g^*_{0,n}(K_{\overline{\mathcal{M}}_{0,n}} + \mathcal{L}).$$

Proof. The proof in [Kaw97, Theorem 2] works in positive characteristic without any change (see also [CTX13, 6.7], [PS09, 8.5], and [KMM94, Section 3]). □

Lemma 5.12 (Stable Reduction Lemma)**.** *Let B be a smooth curve and $f : X \to B$, a flat family of rational curves such that the general fiber is isomorphic to $\mathbb{P}^1$, and a unique singular fiber X_0 over $0 \in B$. Also assume that $f|_{X^*} : (X^* = X \backslash X_0; \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n) \to$*

$B^* = B - \{0\}$ is a flat family of n-pointed stable rational curves sitting in the following commutative diagram

$$\begin{tikzcd} X^* = B^* \times_{\overline{\mathcal{M}}_{0,n}} \overline{\mathcal{U}}_{0,n} \arrow[r] \arrow[d, "f"'] & \overline{\mathcal{U}}_{0,n} \arrow[d] \\ B^* \arrow[r] & \overline{\mathcal{M}}_{0,n} \end{tikzcd} \tag{5.1}$$

Then there exists a unique flat family $\hat{f} : \hat{X} \to B$ of n-pointed stable rational curves satisfying the following commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & \hat{X} = B \times_{\overline{\mathcal{M}}_{0,n}} \overline{\mathcal{U}}_{0,n} \arrow[l, dashed] \arrow[r] \arrow[d, "\hat{f}"'] & \overline{\mathcal{U}}_{0,n} \arrow[d] \\ B & B \arrow[l, "id_B"'] \arrow[r] & \overline{\mathcal{M}}_{0,n} \end{tikzcd} \tag{5.2}$$

where the broken horizontal map is a birational map such that $f^{-1}B^* \cong \hat{f}^{-1}B^*$.

Proof. Since $\overline{\mathcal{M}}_{0,n}$ is a proper scheme, by the valuative criterion of properness, any morphism $B^* \to \overline{\mathcal{M}}_{0,n}$ extends uniquely to a morphism $B \to \overline{\mathcal{M}}_{0,n}$. Now since $\overline{\mathcal{M}}_{0,n}$ has a universal family $\overline{\mathcal{U}}_{0,n}$, the existence of $\hat{f} : \hat{X} \to B$ follows by taking the fiber product. □

5.2 Canonical Bundle Formula

In this section, we work over an algebraically closed field of characteristic $p > 5$ unless stated otherwise.

Canonical bundle formula is one of the main ingredients of the adjunction formula in higher codimension. In this section, we prove the following theorem.

Theorem 5.13. *Let $f : X \to Z$ be a proper surjective morphism, where X is a normal surface and Z is a smooth curve over an algebraically closed field k of char $(k) > 0$. Also assume that $Q = \sum_i Q_i$ is a divisor on Z such that f is smooth over $(Z - Supp(Q))$ with fibers isomorphic to $\mathbb{P}^1$. Let $D = \sum_j d_j P_j$ be a $\mathbb{Q}$-divisor on X, where $d_j = 0$ is allowed, which satisfies the following conditions:*

1. *$(X, D \geq 0)$ is KLT.*

2. *$D = D^h + D^v$, where $D^h = \sum_{f(D_j)=Z} d_j D_j$ and $D^v = \sum_{f(D_j)\neq Z} d_j D_j$. An irreducible component of D^h (resp. D^v) is called horizontal (resp. vertical) component.*

3. *Char* $(k) = p > \frac{2}{\delta}$*, where* δ *is the minimum non-zero coefficient of* D^h*.*

4. $K_X + D \sim_{\mathbb{Q}} f^*(K_Z + M)$ *for some* $\mathbb{Q}$*-Cartier divisor* M *on* Z*.*

Then there exists an effective $\mathbb{Q}$*-divisor* $D_{\mathrm{div}} \geq 0$ *and a semi-ample* $\mathbb{Q}$*-divisor* $D_{\mathrm{mod}} \geq 0$ *on* Z *(as defined in 5.1.2) such that*

$$K_X + D \sim_{\mathbb{Q}} f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

Proof. The sketch of the proof of this formula is given in [CTX13, 6.7]. We include a complete proof following the idea of the proof of [PS09, Theorem 8.1].

First we reduce the problem to the case where all components of D^h are sections. Let D_{i_0} be a horizontal component of D and $Z' \to D_{i_0}$ be the normalization of D_{i_0}. Then $\nu : Z' \to Z$ is a finite surjective morphism of smooth curves. Let X' be the normalization of the component of $X \times_Z Z'$ dominating Z.

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\nu'"'] \arrow[d, "f'"] \\ Z & Z' \arrow[l, "\nu"'] \end{tikzcd} \tag{5.3}$$

Let $k = \deg(\nu : Z' \to Z)$ and l be a general fiber of f. Then

$$k = D_i \cdot l \leq \frac{1}{d_i}(D \cdot l) = \frac{1}{d_i}(-K_X \cdot l) = \frac{2}{d_i} \leq \frac{2}{\delta} < \mathrm{Char}\ (k). \tag{5.4}$$

Therefore, $\nu : Z' \to Z$ is a separable morphism.
Let D' be the log pullback of D under ν', i.e.,

$$K_{X'} + D' = \nu'^*(K_X + D). \tag{5.5}$$

More precisely, we have (by [Kol92, 20.2])

$$D' = \sum_{i,j} d'_{ij} D'_{ij},\ \nu'(D'_{ij}) = D_i,\ d'_{ij} = 1 - (1 - d_i)e_{ij},$$

where e_{ij}'s are the ramification indices along the D'_{ij}'s.

By construction, X dominates Z. Also, since ν is etale over a dense open subset of Z, say, $\nu^{-1}U \to U$, and etale morphisms are stable under base change, $(f' \circ \nu)^{-1}U \to f^{-1}U$ is etale. Thus the ramification locus Λ of ν' does not contain any horizontal divisor of f', i.e., $f'(\Lambda) \neq Z'$. Therefore, D' is boundary near the generic fiber ($\cong \mathbb{P}^1$) of f', i.e., D'^h is effective. We observe that the coefficients of D'^h can be computed by intersecting with a general fiber of $f' : X' \to Z'$, hence they are equal to the coefficients of $D^h \subseteq X$. Thus the condition $p > \frac{2}{\delta}$ remains true for D' on X'.

After finitely many such base changes, let $g : Y \to \tilde{Z}$ be a family such that all of the horizontal components of D_Y are sections of g, where D_Y is the log pullback of D, i.e., $K_Y + D_Y = \psi^*(K_X + D)$.

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "\psi"'] \arrow[d, "g"] \\ Z & \tilde{Z} \arrow[l, "\psi_0"] \end{tikzcd} \tag{5.6}$$

By Lemma 5.12, we get a family of n-pointed stable rational curves $\hat{Y} = \tilde{Z} \times_{\overline{\mathcal{M}}_{0,n}} \overline{\mathcal{U}}_{0,n} \to \tilde{Z}$. Let $\tilde{X}$ be the common resolution of Y and $\hat{Y}$. Let $\hat{X} = \tilde{Z} \times_{\overline{\mathcal{M}}_{0,n}} \mathcal{U}^*_{0,n}$. By the universal property of fiber product, there exists a morphism $\mu : \tilde{X} \to \hat{X}$. Since $\tilde{X}$, $\hat{Y}$, and $\hat{X}$ are all isomorphic $\mathbb{P}^1$-bundles over a dense open subset $U \subseteq \tilde{Z}$, $\mu : \tilde{X} \to \hat{X}$ is birational.

$$\begin{tikzcd} & & \tilde{X} \arrow[dll, bend right, "\pi"'] \arrow[dl, "\lambda"'] \arrow[dd, "\tilde{f}"] \arrow[dr, "\mu"] & & & \\ X \arrow[d, "f"'] & Y \arrow[l, "\psi"'] \arrow[rr, dashed] \arrow[dr, "g"'] & & \hat{X} \arrow[r, dashed] \arrow[dl, "\hat{f}"] \arrow[rr, bend left, "\hat{\phi}"] & \overline{\mathcal{U}}_{0,n} \arrow[r, "\sigma"] \arrow[d, "f_{0,n}"'] & \mathcal{U}^*_{0,n} \arrow[dl, "g_{0,n}"] \\ Z & & \tilde{Z} \arrow[ll, "\psi_0"] \arrow[rr, "\phi_0"] & & \overline{\mathcal{M}}_{0,n} & \end{tikzcd} \tag{5.7}$$

Let $\tilde{D}$ and $\hat{D}$ be $\mathbb{Q}$-divisors on $\tilde{X}$ and $\hat{X}$, respectively, defined by

$$K_{\tilde{X}} + \tilde{D} = \pi^*(K_X + D). \tag{5.8}$$

and

$$K_{\hat{X}} + \hat{D} = \mu_*(K_{\tilde{X}} + \tilde{D}).$$

Since $K_{\tilde{X}} + \tilde{D}$ is a pullback from the base $\tilde{Z}$ (by (5.7)), by the Negativity lemma, we get

$$K_{\tilde{X}} + \tilde{D} = \mu^*(K_{\hat{X}} + \hat{D}). \tag{5.9}$$

Since the definition of the *divisorial part* of the adjunction does not depend on the birational modification of the family (see [PS09, Remark 7.3(ii)] or [Amb99, Remark

3.1]), we will define it with respect to $\hat{f} : \hat{X} \to \tilde{Z}$. First we will show that the $\mathbb{Q}$-divisor $\hat{D}_{\text{mod}}$ on $\tilde{Z}$ is semi-ample.

Since $\hat{\phi}$ is finite and $\mathcal{D}^*$ is horizontal, it follows that $\hat{\phi}^*(\mathcal{D}^*)$ is horizontal too. Since $\hat{D}^h$ is also horizontal, one sees that

$$\hat{D}^h = \hat{\phi}^*(\mathcal{D}^*).$$

From the construction of the $\sigma : \overline{\mathcal{U}}_{0,n} \to \mathcal{U}^*_{0,n}$ we see that $(F, \mathcal{D}^*|_F)$ is log canonical for any fiber F of $g_{0,n} : \mathcal{U}^*_{0,n} \to \overline{\mathcal{M}}_{0,n}$. Since the fibers of $\hat{f} : \hat{X} \to \tilde{Z}$ are isomorphic to the fibers of $g_{0,n}$, $(\hat{F}, \hat{D}^h|_{\hat{F}})$ is also log canonical, where $\hat{F}$ is a fiber of $\hat{f}$. Finally, since $\hat{X}$ is a surface, by inversion of adjunction, $(\hat{X}, \hat{F} + \hat{D}^h)$ is log canonical near $\hat{F}$. Thus we get

$$\hat{D}^v = \hat{f}^* \hat{D}_{\text{div}} \tag{5.10}$$

and

$$K_{\hat{X}} + \hat{D}^h \sim_{\mathbb{Q}} \hat{f}^*(K_{\tilde{Z}} + \hat{D}_{\text{mod}}). \tag{5.11}$$

By (5.11), Lemma 5.11 and [Liu02, Chapter 6, Theorem 4.9 (b) and Example 3.28], we get

$$K_{\hat{X}} + \hat{D}^h - \hat{f}^*(K_{\tilde{Z}} + \phi_0^* \mathcal{L}) = K_{\hat{X}/\tilde{Z}} + \hat{D}^h - \hat{\phi}^* K_{\mathcal{U}^*_{0,n}/\overline{\mathcal{M}}_{0,n}} - \hat{\phi}^*(\mathcal{D}^*) \sim_{\mathbb{Q}} 0. \tag{5.12}$$

Since $\hat{f}$ has connected fibers, by (5.11) and (5.12) and the projection formula for locally free sheaves, we get

$$\hat{D}_{\text{mod}} \sim_{\mathbb{Q}} \phi_0^* \mathcal{L}$$

i.e., $\hat{D}_{\text{mod}}$ is semi-ample.

Now, since $\psi : \tilde{Z} \to Z$ is a composition of finite morphisms of degree strictly less than Char (k), by [Kol13, Corollary 2.43] and [Amb99, Theorem 3.2] (also see [CTX13, 6.6]), we get

$$K_{\tilde{Z}} + \hat{D}_{\text{div}} \sim_{\mathbb{Q}} \psi^*(K_Z + D_{\text{div}}). \tag{5.13}$$

Therefore,

$$\psi^* D_{\text{mod}} \sim_{\mathbb{Q}} \hat{D}_{\text{mod}} \tag{5.14}$$

Since Z and $\tilde{Z}$ are both smooth curves, D_{mod} is semi-ample.

□

5.3 Adjunction Formula

Theorem 5.14. *Let $(X, D \geq 0)$ be a $\mathbb{Q}$-factorial 3-fold log canonical pair such that the coefficients of D are contained in a DCC set $I \subseteq [0,1]$. Let W be a minimal log canonical center of (X, D), and codimension of W is 2. Also assume that X has KLT singularities and $\mathrm{char}(k) > \frac{2}{\delta}$, where δ is the non-zero minimum of the set $D(I)$ (defined in 5.1.1). Then the following hold:*

1. *W is normal.*

2. *There exists effective $\mathbb{Q}$-divisors D_W and M_W on W such that $(K_X + D)|_W \sim_{\mathbb{Q}} K_W + D_W + M_W$. Moreover, if $D = D' + D''$ with D' (resp. D'') the sum of all irreducible components which contain (resp. do not contain) W, then M_W is determined only by the pair (X, D').*

3. *There exists an effective $\mathbb{Q}$-divisor M'_W such that $M'_W \sim_{\mathbb{Q}} M_W$ and the pair $(W, D_W + M'_W)$ is KLT.*

Proof. Normality of W follows from Theorem 4.7.

Since X is $\mathbb{Q}$-Cartier, $(K_X + D)|_W = (K_X + D' + D'')|_W = (K_X + D')|_W + D''|_W$. Thus we may assume that all components of D contain W. Since W is a minimal log canonical center of (X, D) and $\mathrm{codim}_X W = 2$, it does not intersect any other LC center of codimension ≥ 2, by Lemma 4.2. Thus by shrinking X (removing closed subsets of codimension ≥ 2 which do not intersect W) if necessary, we may assume that W is the unique log canonical center of codimension ≥ 2 of (X, D).

Let $f : (X', D') \to (X, D)$ be a $\mathbb{Q}$-factorial DLT model over (X, D) such that

$$K_{X'} + D' = f^*(K_X + D). \tag{5.15}$$

Such f exists by [KK10, 3.1] and [Bir13].

Note that, since X is $\mathbb{Q}$-factorial, the exceptional locus of f supports an effective anti-ample divisor. In particular, all positive dimensional fibers of f are contained in the support of $\lfloor D' \rfloor$.

Let E be an exceptional divisor dominating W. Then E is normal by Proposition 4.3. Write $D' = E + \sum d_i f_*^{-1} D_i$.

By adjunction, we have

$$K_E + D'_E = (K_{X'} + D')|_E = f^*((K_X + D)|_W) \tag{5.16}$$

and (E, D'_E) is DLT, by Proposition 4.3 and the coefficients of D'_E are in the set $D(I)$ by Lemma 5.10.

By Theorem 5.13, there exist $\mathbb{Q}$-divisors $D_W \geq 0$ and $M_W \geq 0$ on W such that

$$K_E + D'_E \sim_{\mathbb{Q}} f|_E^*(K_W + D_W + M_W). \tag{5.17}$$

Since $f|_E : E \to W$ has connected fibers, from (5.16), (5.17) and the projection formula for locally free sheaves, we get

$$(K_X + D)|_W \sim_{\mathbb{Q}} K_W + D_W + M_W. \tag{5.18}$$

Lemma 5.15 given below shows that D_W is independent of the choice of the exceptional divisor f dominating W.

From the definition of D_W, we see that $D_W \geq 0$, since $D'_E \geq 0$. Also, since D_W is independent of the birational modifications (by [PS09, Remark 7.3(ii)]) and W is a minimal LC center, by taking a log resolution of (X', D') and working on the strict transform of E, we see that the coefficients of D_W are strictly less than 1. Thus $\lfloor D_W \rfloor = 0$.

Since M_W is semi-ample and W is a smooth curve, either $M_W = 0$ or M_W is ample. In the later case by Bertini's theorem, there exists an effective $\mathbb{Q}$-divisor $M'_W \sim_{\mathbb{Q}} M_W$ such that $\lfloor M'_W \rfloor = 0$ and $\operatorname{Supp}(M'_W) \cap \operatorname{Supp}(D_W) = \emptyset$. Hence $(W, D_W + M'_W)$ is KLT.

□

Lemma 5.15. *With the same hypothesis as in Theorem 5.14, the divisor $D_W = D_{\text{div}}$ on W is independent of the choice of the exceptional divisor dominating W.*

Proof. Let E_1 and E_2 be two exceptional divisors of f dominating W such that

$$K_{X'} + E_1 + E_2 + \Delta' = f^*(K_X + D), \tag{5.19}$$

where $f : X' \to X$ is the DLT model as above and $D' = E_1 + E_2 + \Delta'$.

By adjunction on E_1, we get

$$K_{E_1} + C + \Delta'_{E_1} = f^*((K_X + D)|_W), \tag{5.20}$$

where C is an irreducible component of $E_1 \cap E_2$.

Adjunction on C gives

$$K_C + \Delta'_C = f^*((K_X + D)|_W). \tag{5.21}$$

Let Q be a point on W, and

$$t = \mathrm{LCT}(E_1, C + \Delta'_{E_1}; f^*Q),\ s = \mathrm{LCT}(C, \Delta'_C; f^*Q|_C).$$

Since C is an irreducible component of $E_1 \cap E_2$ dominating W, it is enough to show that $t = s$. By adjunction, $t \leq s$. So by contradiction, assume that $t < s$.
Since $(E_1, C + \Delta'_{E_1})$ is DLT by Proposition 4.3, $(E_1, C + \Delta'_{E_1} + t' f^*Q)$ is LC outside of $f^{-1}Q$ for any $t' > t$. Thus all NLC centers of $(E_1, C + \Delta'_{E_1} + t' f^*Q)$ appear along $f^{-1}Q$.

The general fiber of $f|_{E_1} : E_1 \to W$ is isomorphic to $\mathbb{P}^1$. Thus degree$((C + \Delta'_{E_1})|_{\mathbb{P}^1}) = 2$ by (5.20). There are two cases depending on whether C intersects the general fiber with degree 1 or 2.

Case I: *C intersects the general fiber with degree* 1. Then there exists a horizontal component C' of Δ'_{E_1}. Let H be an ample divisor on E_1, and F_η, the generic fiber of $f|_{E_1} : E_1 \to W$. Choose $\lambda > 0$ such that

$$(H - \lambda C') \cdot F_\eta = 0.$$

Then $(H - \lambda C')|_{F_\eta} \sim_{\mathbb{Q}} 0$. Thus by [Cor07, 8.3.4], $H \sim_{\mathbb{Q}} \lambda C' - \sum \lambda_i F_i$, where the F_i's are irreducible components of some fibers of f. By adding the pullback of some appropriate divisors from the base to $\lambda C' - \sum \lambda_i F_i$, we may assume that $\lambda_i > 0$ for all i and $\lambda C' - \sum \lambda_i F_i$ is f-ample.

Assume that there exists a point $P \in f^{-1}Q$ but $P \notin C$ such that $(E_1, C + \Delta'_{E_1} + (t + \epsilon) f^*Q)$ is not LC at P, where $0 < \epsilon \ll 1$ such that $t + \epsilon < s$. Then by choosing $0 < \lambda, \lambda_i \ll 1$ we can assume that $(C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i) \geq 0$, $(E_1, C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon) f^*Q)$ still not LC at P, and

$$-\left(K_{E_1}+C+\Delta'_{E_1}-\lambda C'+\sum\lambda_iF_i\right)=-f^*((K_X+D)|_W)+\left(\lambda C'-\sum\lambda_iF_i\right)\quad(5.22)$$

is f-ample.

Then by [Bir13, 8.3], $\mathrm{NKLT}(E_1, C+\Delta'_{E_1}-\lambda C'+\sum\lambda_iF_i+(t+\epsilon)f^*Q)\cap f^{-1}Q$ is connected. Let $R\in C\cap f^{-1}Q$. Then there exists a chain of curves G_i's connecting R and P, and contained in $\mathrm{NKLT}(E_1, C+\Delta'_{E_1}-\lambda C'+\sum\lambda_iF_i+(t+\epsilon)f^*Q)\cap f^{-1}Q$.

Now $\mathrm{NKLT}(E_1, C+\Delta'_{E_1}-\lambda C'+\sum\lambda_iF_i+(t+\epsilon)f^*Q)\subseteq\mathrm{NKLT}(E_1, C+\Delta'_{E_1}+\sum\lambda_iF_i+(t+\epsilon)f^*Q)$. Since we are only concentrating on the NKLT centers along $f^{-1}Q$, we may assume that F_i's are all contained in $f^{-1}Q$. Then by choosing $0<\lambda_i\ll 1$ for all i, such that $t+\epsilon'=t+\epsilon+\max\{\lambda_i\}<s$, we see that $\mathrm{NKLT}(E_1, C+\Delta'_{E_1}+\sum\lambda_iF_i+(t+\epsilon)f^*Q)\subseteq\mathrm{NKLT}(E_1C+\Delta'_{E_1}+(t+\epsilon')f^*Q)$. Thus the curves G_i's are contained in the $\mathrm{NKLT}(E_1, C+\Delta'_{E_1}+(t+\epsilon')f^*Q)$. Hence G_i's are contained in $\mathrm{NLC}(E_1, C+\Delta'_{E_1}+sf^*Q)$. This implies that $(E_1, C+\Delta'_{E_1}+sf^*Q)$ is not LC at $R\in C$. Then by inversion of adjunction, we get a contradiction to the fact that $(C, \Delta'_C+sf^*Q|_C)$ is LC.

Case II: *C intersects the general fiber with degree* 2. In this case, $E_1\cap E_2=C$ and $\Delta'_{E_1}=\Delta'_{E_2}=0$. Since $D\neq 0$ and every component of D contains W, one of the E_i's, say $E_2=f_*^{-1}D_i$, where D_i is an irreducible component of D. Thus in this case, the exceptional divisors of f do not intersect each other. Since X is $\mathbb{Q}$-factorial, the exceptional locus $\mathrm{Ex}(f)$ of $f: X'\to X$ supports an effective anti-ample divisor and hence $\mathrm{Ex}(f)\cap f^{-1}(w)$ is connected for all $w\in W$. Thus f has a unique exceptional divisor in this case and we are done.

□

CHAPTER 6

F-ADJUNCTION

The F-Different was first defined formally by Schwede in [Sch09], and the Different was defined originally by Shokurov. In [Sch09], Schwede proved the equality of the F-Different and the Different for divisors which are Cartier in codimension 2 (see [Sch09, 7.2]) and conjectured that the equality holds in general. We prove this conjecture in this chapter. This equality then gives a second proof (Corollary 6.5) of the Theorem 3.17. Our proof of this equality also closes the gap in Takagi's proof of the equality of restriction of certain generalizations of test ideal sheaves (see [Tak08, Theorem 4.4]), where it is assumed that these two Differents coincide.

6.1 Preliminaries

Let $(X, S+\Delta)$ be a pair, where X is a F-finite normal scheme of pure dimension over a field k of characteristic $p>0$ and $S+\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $(p^e-1)(K_X+S+\Delta)$ is Cartier for some $e>0$. Also assume that S is a reduced Weil divisor, $S \wedge \Delta = 0$ and $\nu : S^n \to S$ is the normalization morphism. Then by [MS12, 4.7] (also see [Sch09, 8.2]), there exists a canonically determined $\mathbb{Q}$-divisor $\Delta_{S^n} \geq 0$ on S^n such that $\nu^*(K_X+S+\Delta) \sim_{\mathbb{Q}} K_{S^n}+\Delta_{S^n}$.

Definition 6.1. The divisor $\Delta_{S^n} \geq 0$ defined above is called the F-Different and it is denoted by $F\text{-Diff}_{S^n}(\Delta)$.

Let $(X, S+\Delta)$ be a pair as above. Then following the construction of [Kol92, Chapter 16] or [Kol13, Definition 4.2], we see that there exists a canonically determined $\mathbb{Q}$-divisor $\Delta'_{S^n} \geq 0$ on S^n such that $\nu^*(K_X+S+\Delta) \sim_{\mathbb{Q}} K_{S^n}+\Delta'_{S^n}$.

Definition 6.2. The divisor Δ'_{S^n} defined above is called the Different and it is denoted by $\mathrm{Diff}_{S^n}(\Delta)$.

We have given an example of the Different in Chapter 1, Example 2.4.1. The following example of F-Different is taken from [BS13, Example 7.1.2].

6.1.1 Example

Let $S = k[x,y,z]$, $\mathrm{char}(k) = p > 2$, and $R = k[x,y,z]/(xy - z^2)$. Then $X = \operatorname{Spec} R$ is a cone over a degree 2 rational curve $C \subseteq \mathbb{P}^3$. Let $D = V(x,z)$ be a ruling of the cone X. We will compute the F-Different for the pair (X, D). $F^e_* S$ is a free S module with a basis $\{F^e_*(x^{\lambda_1} y^{\lambda_2} z^{\lambda_3}) : 0 \leq \lambda_i \leq p^e - 1 \text{for all } i\}$. Let $\Phi_S \in \mathrm{Hom}_S(F^e_* S, S)$ be a map defined by

$$\Phi_S\left(F^e_*(x^{\lambda_1} y^{\lambda_2} z^{\lambda_3})\right) = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2 = \lambda_3 = p^e - 1 \\ 0 & \text{otherwise.} \end{cases}$$

By Fedder's lemma [BS13, Lemma 6.2.1], the map $\Psi(F^e_* \_) = \Phi(F^e_*((xy - z^2)^{p^e-1} \cdot \_))$ induces a $F^e_* R$-module generator of $\mathrm{Hom}_R(F^e_* R, R)$. Observe that $\mathcal{O}_X(-2nD) = (x^n)$.

Now consider the map

$$\varphi_X(F^e_* \_) = \Psi(F^e_*(x^{\frac{p^e-1}{2}} \cdot \_)) = \Phi_S(F^e_*(x^{\frac{p^e-1}{2}} (xy - z^2)^{p^e-1} \cdot \_)).$$

We see that φ_X corresponds to the divisor $(p^e - 1)D$. It is easy to see that φ_X is compatible with respect to D. Thus we obtain φ_D. To compute the divisor associated to φ_D, we need to read off the term containing $x^{p^e-1} z^{p^e-1}$ in

$$\left(x^{\frac{p^e-1}{2}}\right)(xy - z^2)^{p^e-1} = x^{\frac{3(p^e-1)}{2}} y^{p^e-1} + \cdots + \binom{p^e-1}{\frac{p^e-1}{2}} xz^{p^e-1} y^{\frac{p^e-1}{2}} + \cdots$$

The reason this works is because the map $\Phi_S(F^e_*(x^{p^e-1} z^{p^e-1} \cdot \_))$ induces the generator of $\mathrm{Hom}_{\mathcal{O}_D}(F^e_* \mathcal{O}_D, \mathcal{O}_D)$ as $F^e_* \mathcal{O}_D$-module. But $\binom{p^e-1}{\frac{p^e-1}{2}} \neq 0 \pmod p$ and so if $\Phi_D : F^e_* k[y] \to k[y]$ is the map generating $\mathrm{Hom}_{\mathcal{O}_D}(F^e_* \mathcal{O}_D, \mathcal{O}_D)$, then φ_D (which is restriction of φ_X to D) is defined by the rule

$$\varphi_D(F^e_* \_) = \Phi_D\left(F^e_* y^{\frac{p^e-1}{2}} \cdot \_\right)$$

at least up to multiplication by an element element of k. Thus the associated divisor is

$$\Delta_D = \frac{1}{p^e - 1} \operatorname{div}\left(y^{\frac{p^e-1}{2}}\right) = \frac{1}{2} \operatorname{div}(y).$$

The divisor Δ_D is the F-Different on D which appears in the F-adjunction of the pair (X, D), i.e., $(K_X + D)|_D \sim_{\mathbb{Q}} K_D + \Delta_D$.

Remark 6.3. For the general theory of F-Adjunction, see [Sch09] and [BS13]. We follow the definitions of the ideals $\tau_b(X; \Delta)$ and $\tau_b(X, \not\subseteq Q; \Delta)$ as in [BSTZ10].

6.2 F-Different Is Not Different from the Different

In this section, we show that the F-Different coincides with the Different.

Theorem 6.4. *Let $(X, S + \Delta \geq 0)$ be a pair, where X is a F-finite normal excellent scheme of pure dimension over a field k of characteristic $p > 0$ and $S + \Delta \geq 0$ is a $\mathbb{Q}$-divisor on X such that $(p^e - 1)(K_X + S + \Delta)$ is Cartier for some $e > 0$. Also assume that S is a reduced Weil divisor and $S \wedge \Delta = 0$. Then the F-Different, F-$\mathrm{Diff}_{S^n}(\Delta)$ is equal to the Different, $\mathrm{Diff}_{S^n}(\Delta)$, i.e., F-$\mathrm{Diff}_{S^n}(\Delta) = \mathrm{Diff}_{S^n}(\Delta)$, where $S^n \to S$ is the normalization morphism.*

Proof. First observe that F-$\mathrm{Diff}_{S^n}(\Delta)$ and $\mathrm{Diff}_{S^n}(\Delta)$ are both divisors on S^n, so it is enough to prove that they are equal at all codimension 1 points of S^n. Since the codimension 1 points of S^n lie over the condimension 1 points S, by localizing X at a codimension 1 point of S we can assume that X is an excellent surface.

Since $S + \Delta \geq 0$, by [BS13, 4.1], it induces a map

$$\varphi : F^e_* \mathscr{L} \to \mathcal{O}_X, \tag{6.1}$$

where

$$\mathscr{L} = \mathcal{O}_X((1 - p^e)(K_X + S + \Delta)) \text{ is a line bundle.}$$

Let $\pi : Y \to X$ be a log resolution of $(X, S + \Delta)$ (log resolution exists for excellent surfaces by [Abh65] and [Hir84], also see [Lip78]) such that $\pi^{-1}_* S = \widetilde{S}$ is smooth and

$$K_Y + \widetilde{S} + \Delta_Y = \pi^*(K_X + S + \Delta). \tag{6.2}$$

Then by [BS13, 7.2.1], we have a morphism

$$\varphi_Y : F^e_* \pi^* \mathscr{L} \to \mathscr{K}(Y) \tag{6.3}$$

where $\mathscr{K}(Y)$ is the constant sheaf of rational functions on Y, such that φ_Y agrees with φ wherever π is an isomorphism.

Let $\Delta_Y = \widetilde{\Delta} + \Sigma a_i E_i$, where $\widetilde{\Delta}$ is the strict transform of Δ and E_i's are the exceptional divisors of π. Then we can factor φ_Y in the following way

$$F^e_*\mathcal{O}_Y((1-p^e)(K_Y+\widetilde{S}+\widetilde{\Delta}+\Sigma a_i E_i)) \subseteq F^e_*\mathcal{O}_Y((1-p^e)(K_Y+\Sigma a_i E_i)) \to \mathscr{K}(Y). \tag{6.4}$$

Let $N \geq 0$ be a sufficiently large Cartier divisor on Y such that

$$\varphi_Y(F^e_*\mathcal{O}_Y((1-p^e)(K_Y+\Sigma a_i E_i))) \subseteq \mathcal{O}_Y(N). \tag{6.5}$$

Then from (6.4), we have

$$\varphi_Y : F^e_*\pi^*\mathscr{L} \to \mathcal{O}_Y(N). \tag{6.6}$$

We see that $\widetilde{S}$ is φ_Y-compatible in the following way

$$F^e_*\mathcal{O}_Y((1-p^e)(K_Y+\widetilde{S}+\Delta_Y)-\widetilde{S}) \subseteq F^e_*\mathcal{O}_Y((1-p^e)(K_Y+\Sigma a_i E_i)-p^e\widetilde{S}) \to \mathcal{O}_Y(N-\widetilde{S}). \tag{6.7}$$

Thus we get the following induced morphism on $\widetilde{S}$ ($cf.$ [BS13, 6.0.3])

$$\overline{\varphi}_Y : F^e_*\pi^*\mathscr{L}|_{\widetilde{S}} \to \mathcal{O}_{\widetilde{S}}(N|_{\widetilde{S}}). \tag{6.8}$$

Since $\pi|_{\widetilde{S}} : \widetilde{S} \to S$ is the normalization morphism, by [Kol13, 4.7], we have

$$K_{\widetilde{S}} + \Delta_Y|_{\widetilde{S}} \sim_{\mathbb{Q}} K_{\widetilde{S}} + \operatorname{Diff}_{\widetilde{S}}(\Delta) \text{ and } \operatorname{Diff}_{\widetilde{S}}(\Delta) = \Delta_Y|_{\widetilde{S}}. \tag{6.9}$$

Since $\operatorname{Diff}_{\widetilde{S}}(\Delta) \geq 0$ as $\Delta \geq 0$, from (6.9), we get that $\Delta_Y|_{\widetilde{S}} \geq 0$. This implies that $\overline{\varphi}_Y(F^e_*\pi^*\mathscr{L}|_{\widetilde{S}}) \subseteq \mathcal{O}_{\widetilde{S}}$, since we have the following factorization of $\overline{\varphi}_Y$:

$$F^e_*\pi^*\mathscr{L}|_{\widetilde{S}} = F^e_*\mathcal{O}_{\widetilde{S}}((1-p^e)(K_{\widetilde{S}}+\Delta_Y|_{\widetilde{S}})) \subseteq F^e_*\mathcal{O}_{\widetilde{S}}((1-p^e)(K_{\widetilde{S}})) \to \mathcal{O}_{\widetilde{S}}. \tag{6.10}$$

Thus we get the following commutative diagram:

$$\begin{tikzcd}
F^e_*\mathscr{L}|_{S^n} \arrow[r] \arrow[d, equal] & \mathcal{O}_{S^n} \arrow[d, equal] \\
F^e_*\pi^*\mathscr{L}|_{\widetilde{S}} \arrow[r, "\overline{\varphi}_Y"] & \mathcal{O}_{\widetilde{S}}
\end{tikzcd}$$

since $S^n = \widetilde{S}$.

Now from the commutative diagram above, we get that

$$F\text{-Diff}_{S^n}(\Delta) = F\text{-Diff}_{\widetilde{S}}(\Delta_Y).$$

Since $\Delta_Y|_{\widetilde{S}} \geq 0$, by working locally on a neighborhood of $\widetilde{S}$, we can assume that $\Delta_Y \geq 0$. Since Y is smooth and $\widetilde{S} + \Delta_Y \geq 0$ has simple normal crossing support, by [Sch09, 7.2], $F\text{-Diff}_{\widetilde{S}}(\Delta_Y) = \Delta_Y|_{\widetilde{S}}$. But $\Delta_Y|_{\widetilde{S}} = \text{Diff}_{S^n}(\Delta)$ by (6.9). Therefore, $F\text{-Diff}_{S^n}(\Delta) = \text{Diff}_{S^n}(\Delta)$. □

Corollary 6.5. *Let $(X, S+B)$ be a pair, where X is a normal variety, $S+B \geq 0$ is a $\mathbb{Q}$-divisor, $K_X + S + B$ is $\mathbb{Q}$-Cartier, and $\lfloor S+B \rfloor = S$ is reduced and irreducible. Let $\nu : S^n \to S$ be the normalization morphism, write $(K_X + S + B)|_{S^n} = K_{S^n} + B_{S^n}$. If (S^n, B_{S^n}) is strongly F-regular, then S is normal and $(X, S+B)$ is purely F-regular near S.*

Proof. The question is local on the base, thus we can assume that $X = \operatorname{Spec} R$. Let D' be an effective Weil divisor on X such that $D' - K_X$ is Cartier. Let S' be another effective Weil divisor on X such that $S' \sim S$ but S' does not contain S. Let $D = \frac{S'+B+D'}{p^e-1} \geq 0$ for $e \gg 0$. D is an effective $\mathbb{Q}$-Cartier divisor. Then $K_X + S + B + D$ is a $\mathbb{Q}$-Cartier divisor with index not divisible p and $\lfloor S+B+D \rfloor = S$. Then index of $K_{S^n} + B_{S^n} + D|_{S^n}$ is also not divisible by p, where $(K_X + S + B + D)|_{S^n} = K_{S^n} + B_{S^n} + D|_{S^n}$. Choosing $e \gg 0$, we can assume that $(S^n, B_{S^n} + D|_{S^n})$ is strongly F-regular. Therefore, we are reduced to the case where the indexes of $K_X + S + B$ and $K_{S^n} + B_{S^n}$ are both not divisible by p.

Since by Theorem 6.4, $B_{S^n} = F\text{-Diff}_{S^n}(B)$ and (S^n, B_{S^n}) is strongly F-regular, (S^n, B_{S^n}) has no proper nontrivial center of F-purity by [Sch10, 4.6]. Let J be the conductor of the normalization $S^n \to S$. Then by [Sch09, 8.2], J is F-compatible with respect to (S^n, B_{S^n}). If $J \neq A$, then by [Sch10, 4.10] and [Sch10, 4.8], we arrive at a contradiction. Thus $S \cong S^n$, i.e., S is normal.

Let Q be the generic point of S. Then by [BSTZ10, 3.15], $\tau_b(X, \not\subseteq Q; S+B)|_S = \tau_b(S; B_S)$. Since $K_S + B_S$ is $\mathbb{Q}$-Cartier, $\tau_b(S; B_S) = \tau_b(S; B_S)$ by [BSTZ10, 3.7]. Therefore, $\tau_b(X, \not\subseteq Q; S+B)|_S = \tau_b(S; B_S)$. Since (S, B_S) is strongly F-regular, $\tau_b(S; B_S) = \mathcal{O}_S$. Thus $\tau_b(X, \not\subseteq Q; S+B)|_S = \mathcal{O}_S$, hence $(X, S+B)$ is purely F-regular near S.

□

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